Category theory in a $(\mathcal{E}, \mathcal{M})$-category

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We propose a generalization of the category of categories which is based on a factorization system $(\mathcal{E}, \mathcal{M})$, playing the role of the comprehensive factorization system on **Cat** [5].

Let $\mathcal{C}$ be any finitely complete category with a factorization system on it. Let us call the arrows in $\mathcal{E}$ and in $\mathcal{M}$ "final maps" and "discrete maps" respectively; and let us denote by $\downarrow(-) : \mathcal{C}/X \to \mathcal{M}/X$ the reflection in discrete maps over X. If the map $m : M \to X$ in $\mathcal{M}/X$ is the discrete reflection $\downarrow x$ of an "object" $x : 1 \to X$ (i.e. if M has a final "object" $e : 1 \to M$), we say that m is a "principal map". (In **Cat**, the principal maps over X are the discrete fibrations X/x, corresponding to the representable presheaves).

If a map $p : P \to X$ in $\mathcal{C}$ has a reflection in principal maps over X, this is called a "colimit" of p. If the discrete reflection $\downarrow p$ is already principal, then it is an "absolute" colimit of p [3, 4]. If the pullback $f^* \downarrow y$ of the principal map on $y : 1 \to Y$ along $f : X \to Y$ is itself principal, we say that the corresponding final object $1 \to f^* \downarrow y$ is a "universal arrow" from f to y. One can also naturally define categorical concepts such as "dense" or "full and faithful" maps.

This very general context seems best suited to einlighten some basic classical facts of category theory, such as the following:

1. If $e : P \to X$ is a final map in $\mathcal{C}$, then for any map $f : X \to Y$ the colimits of f and $f \circ e$ are the same (either existing if the other one does).

2. If a final map $e : P \to X$ has a colimit, then this is absolute and is (the reflection of) a final object of X.

3. A map $f : X \to Y$ admits a universal arrow to an object y of Y, if and only if $f^* \downarrow y \in \mathcal{M}/X$ has a colimit which is preserved by f itself.

4. If a map $f : X \to Y$ admits a universal arrow to any object of Y, then it preserves colimits.

We say that a map $\mathbf{y} : X \to \mathcal{P}X$ presents $\mathcal{P}X$ as a "power object" of X if it induces an equivalence between the category of principal maps over Y and the category of discrete maps over X. If this is the case, the "Yoneda map" $\mathbf{y}$ is full and faithful and dense and the following hold:

1. A map $e : P \to X$ is final *if and only if* for any map $f : X \to Y$ the colimits of f and $f \circ e$ are the same (either existing if the other one does).

2. A colimit of $p : P \to X$ is absolute *if and only if* it is preserved by any map $f : X \to Y$.

3. The discrete reflection of a map $p : P \to X$ can be obtained as the colimit of $\mathbf{y} \circ p$ in $\mathcal{P}X$ (or more precisely, as its correspective via the above equivalence).

Axioms regarding a duality functor $(-)' : \mathcal{C} \to \mathcal{C}$, exponentiability and an arrow object may be added to obtain a more faithful abstraction of **Cat**. E.g., if the object $\Omega = \mathcal{P}1$ of "internal sets" exists, and if $\mathcal{P}X = \Omega^{X'}$, then X has a "hom map" $h : X \times X' \to \Omega$. An arrow object for $\mathcal{C}$ is a bipointed object $s, t : 1 \to 2$ such that the factorization system and its dual are generated by t and s respectively. In this case, to an "arrow" $f : 2 \to X$ of X there corresponds a morphism in $\mathcal{M}/X$ between the principal maps on its domain and codomain; and the map $X \to 1$ is discrete if and only if X is discrete in the sense of [1, 2], since $2 \to 1$ is itself final.

As a further instance of the theory, we consider the category of posets with the usual "cofinal" mappings and the inclusions of lower sets as discrete maps, wherein Ω is the two-elements truth values poset.

References

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