Combinatorial model categories

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A model category $\mathcal{K}$ is equipped with three classes of morphisms $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ called cofibrations, fibrations and weak equivalences. Cofibrations (fibrations) which are weak equivalences are called trivial cofibrations (fibrations). A model category is called *combinatorial* if $\mathcal{K}$ is locally presentable and both cofibrations and trivial cofibrations are generated by a set of morphisms. This concept was introduced by J. H. Smith approximately 10 years ago. Unfortunately, he has not yet published proofs of a number of important results he announced. A striking example is the fact that the colimit closure of simplices in topological spaces form a locally presentable category and thus a combinatorial model category.

I will supply some missing proofs and explain what I do not understand yet. In particular, it will be shown that a colimit closure of a small full subcategory in a fibre-small topological category is always locally presentable, which covers the just mentioned case of simplices. Another proved result is that $\mathcal{W}$ is an accessible subcategory of $\mathcal{K}^{\rightarrow}$ and that, which is a new result, $\mathcal{C}$ is a full image of an accessible functor into $\mathcal{K}^{\rightarrow}$. J. H. Smith also announced that, for a given $\mathcal{C}$ generated by a set of morphisms, there is the smallest $\mathcal{W}$ making $\mathcal{K}$ a combinatorial model category. Independently, we came to the same idea with W. Tholen in 2000 and showed that it is true under set-theoretical Vopěnka's principle. Independently and in the same time, D.-C. Cisinski introduced this idea as well and proved it, without any set theory, in the special case when $\mathcal{K}$ is a Grothendieck topos and $\mathcal{C}$ is the class of all monomorphisms. A current status of this problem will be discussed.