Weak complicial sets and internal quasi-categories

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It is well known that we may represent (strict) ω-categories as simplicial sets, via Street's ω-categorical nerve construction [2]. What may be less well known, is that we may extend Street's nerve functor to one which has been shown to be fully-faithful (Verity [3]). This is achieved by augmenting each simplicial set in the codomain of this functor with a specified subset of *thin simplices* and restricting our attention to those simplicial maps that preserve the property of thinness, to obtain the category of *stratified* simplicial sets. Under this representation, we gain an equivalence between the category of (strict) ω-categories and a certain category of *complicial sets* which was originally defined and studied by Roberts [1].

Complicial sets are characterised, amongst the stratified simplicial sets, by a *strict horn filler* condition under which those simplicial horns that satisfy a certain admissibility criterion have *unique* fillers by thin simplices. By weakening this, to only insist on existence (rather than unique existence) of thin fillers for such horns, we obtain a larger class of *weak complicial sets.* Aside from extending the class of complicial sets, and thus the category of (strict) ω-categories, this class includes all Kan complexes and all quasi-categories (weak Kan complexes). Furthermore, weak complicial sets may be shown to be the fibrant objects for a Quillen model structure on the category of stratified simplicial sets, which extends both the canonical model structure on simplicial sets and Joyal's quasi-categorical model structure (see [4]).

Finally, it is also known that we may canonically enrich the category of weak complicial sets over itself, in a manner that corresponds directly to our enrichment of the category of 2-categories to a Gray-category. By taking the *homotopy coherent nerve* of this structure, we can show that the totality of all weak complicial sets can itself be regarded as a richly structured (large) weak complicial set of homomorphisms and higher strong transformations [5]. Taken as a whole, these results lead us to regard the theory of weak complicial sets as a full weak ω-category theory, modelled upon a simplicial rather than a globular geometry.

While the works cited above provide us with quite a bit of information about the homotopy theory of weak complicial sets, they do little to illuminate their (weak) category theory. To that latter end we follow the path taken by the founders of 2-category theory who observed, early on, that much of the fundamental category theory of their discipline arose directly from the observation that 2-categories could be regarded as categories internal to or enriched over <u>Cat</u>, the category of all small categories. It turns out that we may employ an analogous approach to weak complicial sets, representing them as certain *quasi-categories internal to* the category of weak complicial sets itself.

In this talk, we plan to introduce just enough weak complicial set theory to allow us to describe the construction of this quasi-categorical representation. This will lead naturally to a discussion of the intuitions that underlie the development of the resulting internal quasi-category theory. Finally, we apply this work to formulating a conjectured coherence result for weak complicial sets.

References

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