# Analysis and Enriched Category Theory

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## Introduction

About 30 years ago, Lavere had an ingenious idea to consider metric spaces as enriched categories([3]). As we have also seen at this conference, Tholen, Hofman, and Clementino have extended this idea to consider topological spaces([2]). What I'd like to do today is consider the possibility of going in the opposite direction: if we can give objective meaning to metric spaces and topological spaces, can we do something for normed spaces? If we can, the reward for such an endeavour would be great: there is a vast literature on theorems in normed spaces, and one could do as Lawvere, Tholen, Hofman and Clementino have done: apply category theory to analysis, and analysis to category theory.

## I: Metric Spaces as Enriched Categories

Let us begin by reviewing how Lawvere found the connection between metric spaces and enriched categories. He noticed that there was a similarity between the triangle inequality for a metric space:

$$d(x, y) + d(y, z) \geq d(x, z)$$

and the composition law for categories:

$$\mathbf{C}(a, b) \times \mathbf{C}(b, c) \to \mathbf{C}(a, c)$$

The connection is that they are both forms of enriched categories. The first is over the monoidal category $([0, \infty], \geq, +, 0)$, the second over $(\mathbf{set}, \times, 1)$.

However, it is important to note that a category enriched over $([0,\infty],\geq,+,0)$ is not exactly a metric space in the classical sense. Specifically, it is a set $X$, together with a function $d : X \times X \to [0,\infty]$ such that:

1. $d(x,x) = 0$ (identity arrow)
2. $d(x,y) + d(y,z) \geq d(x,z)$ (composition arrow)

It differs from the classical metric spaces in three ways:

1. $d(x,y) = 0 \not\Rightarrow x = y$ (isomorphic objects are not neccesarily equal)
2. $d$ can take the value $\infty$ (completeness of the base category)
3. $d(x,y) \not\Rightarrow d(y,x)$ (non-symmetry)

In his paper, Lawvere gives good reasons why this version of metric space should be preferred to the classical version. As an example, if one wishes one's metric to be the amount of work it takes to walk around a mountainous region, it should be non-symmetric. In addition, the fact that Lawvere's metric spaces are non-symmetric will be important for us shortly.

# II: Normed Spaces

Now we move on to normed spaces. Recall that a normed vector space is a $\mathbb{C}$ or $\mathbb{R}$ vector space $A$, together with a function $\|\cdot\| : A \to [0,\infty]$ such that

1. $\|0\| = 0$
2. $\|a\| + \|b\| \geq \|a+b\|$
3. $\|\alpha a\| = |\alpha|\|a\|$

Note that we have expanded the definition slightly, by allowing the value $\infty$ and only requiring a semi-norm, just as Lawvere does with his metric spaces.

However, even this is too much information to attempt to directly categorify, so let us instead begin with a slightly weaker notion, namely that of a normed abelian group. This is an abelian group $G$, together with a norm $\|\cdot\| : G \to [0,\infty]$ which satisfies (1) and (2) for normed vector spaces (since it is not a vector space, it obviously cannot satisfy (3)).

We should note a few points about normed abelian groups. First of all, with the notable recent exception of Marco Grandis' work on normed homology ([3]) (where instead of a sequence of abelian groups, one assigns a sequence of normed abelian groups to a space), they have not been considered much in the literature. However, one reason that they may not have been considered is that the metric they define $d(a, b) := \|b - a\|$ is not a metric in the classical sense - it lacks symmetry of distance. Of course, this was one of the key points about Lawvere's metric spaces, and one can show that the metric as defined above is a Lawvere metric. Thus the notion of a normed abelian group is much easier to work with and understand if one has accepted the Lawvere definition of a metric space over the classical one.

One should also note that the scalar invariance axiom (3) for a normed vector space can be recovered from the two axioms (1, 2), for a normed abelian group. How does one do this? Recall that if one equips **ab** with the usual tensor product $\otimes$, then a ring $R$ is the same thing as a one-object **ab**-category, and a module $M$ over $R$ is the same thing as an **ab**-functor $M : R \to$ **ab**. Now, one can also equip the category **normab** of normed abelian groups (maps as homomorphic contractions) with a modification of the projective tensor product of Banach spaces. Then a normed ring $R$ would be a one-object **normab**-category, and a normed module $M$ over a normed ring $R$ would be a **normab**-functor $M : R \to$ **normab**. If one works this through, the extra axiom one gets for a normed module is $\|\alpha a\| \leq |\alpha|\|a\|$. However, and this is the key point, over the normed ring $\mathbb{R}$ or $\mathbb{C}$, this inequality axiom for a normed module actually *implies* the usual equality axiom. Thus, the usual scalar invariance axiom for a normed vector space can be recovered from the axioms for a normed abelian group. Thus the fundamental concept one needs to work with is that of normed abelian group.

So, now us let see if we can categorify the notion of normed abelian group. To do this, we note that $\geq$ is really the arrows in $[0, \infty]$, the plus is tensor, and 0 is the identity I. Thus the axioms for a norm on an abelian group become, if we write $\|\cdot\|$ as $N$:

1. $N(a) \otimes N(b) \to N(a + b)$
2. $I \to N(0)$

We can now recognize these two axioms as the neccesary comparison arrows for a lax monoidal functor. In other words, if we make the abelian group $G$ into a discrete group, with $\otimes$ as $+$, and $I$ as 0, then a lax monoidal functor from $G$ to $[0, \infty]$ is *exactly* a norm on $G$.

Of course, there is one additional piece of information that is not considered in this analysis, namely the fact that $G$ is not just a monoid, but is actually a group. To make use of this, we note that $G$ considered as a monoidal category is actually compact closed, with $* = -$. Thus have that a normed abelian group consists of an compact closed category $G$, together with a lax monoidal functor to $[0, \infty]$. We thus generalize to make the following definition:

**Definition** For $V$ a monoidal category, a **normed space over V** is a compact closed category $C$, together with a lax monoidal functor $N : C \to V$.

Before we proceed, let us determine if this makes sense. We claim that as metric space is to category enriched over $V$, so normed abelian group is to normed space over $V$. However, as we have seen above, every normed abelian group defines a metric space via $d(a, b) := N(b - a)$. So, by analogy, if this were to make sense, every normed space over $V$ should define a category enriched over $V$.

Of course, this is exactly the case. If $N : C \to V$ is a normed space over $V$, then since $C$ is compact closed, it is closed, and so enriched over itself. Since $N$ is a lax monoidal functor, it preserves enrichment, and so just as $d(a, b) := N(a - b)$ defines a metric space, so $C(a, b) := N([a, b])$ defines a enrichment of $C$ over $V$. Thus the analogy does hold at this higher level.

Let us see if what further examples there are of normed spaces over a monoidal category $V$.

1. Recall that a category enriched over $\mathbf{2} := (0 \leq 1, \wedge, 1)$ is a partially ordered set. Thus it works out very well that if $G$ is a abelian group, then a lax monoidal functor from $G$ to $\mathbf{2}$ is exactly an ordered abelian group. So ordered abelian groups are examples of normed spaces over $\mathbf{2}$.

2. Every compact closed category $C$ is normed over **set**, via $N(-) :=$

$C(I, -)$ (the "elements of" construction). For example, the norm of a finite vector space is its underlying set. If we generalize to merely closed categories, then the norm of a Banach space would be its unit ball, which agrees with one's intuition from functional analysis.

3. Recall that a Tannakian category is an abelian compact closed category with an exact faithful monoidal functor to **vec**. Thus Tannakian categories are examples are categories normed over **vec**.

# Conclusion

At this point, the theory seemed to be fairly well-founded, and I began showing that a number of constructions with normed abelian groups are the exact same as constructions one can do with ordered abelian groups, and that they can be extended to the examples mentioned above.

However, upon looking at Lawvere's paper again, I noticed something that I had missed earlier: he had already anticipated the need to objectify norms, but in a slightly different way. Specifically, he mentions that while the "first approximation" is metric spaces and $V$-categories, the "second approximation" consists of normed abelian groups and $V$-compact closed $V$-categories. In other words, Lawvere's objectification of normed abelian group is $V$-compact closed $V$-categories.

The next step, then, is to determine the relationship between our stated objectification of normed abelian groups (compact closed category with lax monoidal functor to $V$) and Lawvere's ($V$-compact closed $V$-categories). It it is my belief that these two categories stand in adjoint relationship to one another, each representing different aspects of normed spaces: the first its "space with a norm" aspects, the second with its metric space aspects.

At this point, the theory is still in its infancy, and still requires much work to be as succesful as the metric space theory of Lawvere, or the topological space theory of Hofman/Clementino/Tholen. However, there is evidence to suggest that it is on the right track, and as mentioned at the beginning, the rewards for a succesful theory would be great.

References

[1] M. Grandis, *Normed combinatorial homology and noncommutative tori*, Theory Appl. Categ. 13 (2004), No. 7, 114-128.

[2] D. Hofman, M. Clementino, W. Tholen, *One setting for all: metric, topology, uniformity, approach structure*, Applied Categorical Structures 12 (2004), 127-154.

[3] F.W. Lawvere, *Metric spaces, generalized logic, and closed categories*, TAC Reprints (http://www.tac.mta.ca/tac/reprints/).