Extended real number object in the bornological topos

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Acknowledgements and references

To Bill Lawvere

1973. F. W. Lawvere. "Metric spaces, generalized logics, and closed categories". *Rend. Sem. Mat. Fis. di Milano*, 43, pp. 135-166.

$\overline{\mathbb{R}}^{+} = [0, \infty]$ as distance-norm-recipient.

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The study of the bornological topos was encouraged.

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Extended real number object in an elementary topos.

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First steps on the bornological topos.

2000. L. Español, L. Lambán. "On bornologies, locales and toposes of M-sets". *J. Pure Appl. Algebra*, 176, pp. 113-125.

An improvement of a part of the Lambán PhD.

...it follows, and ...to be continued ...

Notations for $MSet, \ M = Set(\mathbb{N}, \mathbb{N})$

We consider M-sets E with a right action denoted by composition:

$$x \circ f \in E, \ \ x \in E, \ f \in M.$$

$\alpha : E \to L$ is equivariant if $\alpha(x \circ f) = \alpha(x) \circ f$.

M is an M-subset and morphisms $M \to E$ represent elements of E. Morphisms $1 \to E$ represent fixed elements of E: set $\Gamma(E)$. Each set X is a trivial M-set $\Delta(X)$ (any element is fixed).

If S is an M-subset of E and $x \in E$ then

$$\langle x \in S \rangle = \{ f \in M; x \circ f \in S \}$$

is an ideal (M-subset of M).

. . .

The set Ω of all ideals I of M, with the action $\langle x \in I \rangle$, is the subobject classifier of *MSet*.

MSet is cartesian closed, $(-) \times E \dashv (-)^E : MSet \to MSet$:

- $\theta : P \to L^E$, $\theta(p)(f, x) = \xi(p \circ f, x)$
- $\xi : P \times E \to L$, $\xi(p, x) = \theta(p)(id, x)$

- *evaluation morphism*: $L^E \times E \to L$, $ev(\xi, x) = \xi(id, x)$

- *membership relation*: $\langle x \in_M \xi \rangle = \{f \in M; \xi(f, x \circ f) = M\}$.

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$\zeta : E \to \Omega^L,\ \ \zeta(x) = \{y; \varphi(x, y)\}$ adjoint of $\varphi : E \times L \to \Omega$.

$Im : L^E \times \Omega^E \to \Omega^L,\ \ Im(\alpha, H) = \{y; \exists x, x \in H\ \wedge\ \alpha(x) = y\}$.

M-sets and covariant analysis

Sequence spaces as M-sets:
An M-set E is *separated* if the unit $E \to \Gamma(E)^{\mathbb{N}}$ is mono.
$E \cong \Sigma(X)$, bounded sequences in a bornological space X
(we only consider sequential bornologies).

The full and faithful $\Sigma : Born \to MSet$ preserves exponentials.

Examples of non-separated M-sets:

- Ω, with $Cont \dashv Ext : \Omega \to \mathcal{P}(\mathbb{N})$, $Ext \circ Cont = id$
- $\overline{\mathbb{R}}^+_m = \{\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}^+ ; \mu$ monotone, $\mu(\emptyset) = 0\}$
- Outer measures (add countably subadditive).

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Discrete measures does't give M-sets.

Extended real number M-set, $\overline{\mathbb{R}}_m \subseteq \Omega^{\mathbb{Q}}$

Parts of $\mathbb{Q}$ in *MSet*:

Because $\mathbb{Q}$ is trivial in *MSet*, elements of $\Omega^{\mathbb{Q}}$ are (equivalently):

- $a : \mathbb{Q} \to \Omega \subseteq \mathcal{P}(M)$, $\quad a(x) = \{f \in M; x \in \alpha(f)\}$
- $\alpha : M \to \mathcal{P}(\mathbb{Q})^{op}$ monotone, $\alpha(f) = \{x \in \mathbb{Q}; f \in a(x)\}$
  $f \leq g$ if $Im(f) \subseteq Im(g)$ is a preorder in M
- $\mu : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Q})^{op}$ monotone, $\mu(\emptyset) = \mathbb{Q}$,
  $\mu(A) = \alpha(f)$, $A = Im(f)$

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Actions:

- $(a \circ f)(x) = \langle f \in a(x) \rangle, \quad (\alpha \circ f)(g) = \alpha(f \circ g)$
- $(\mu \circ f)(A) = \mu(f(A))$

...Upper cuts of $\mathbb{Q}$ in *MSet*

$\overline{\mathbb{R}}_m \subseteq \Omega^{\mathbb{Q}}$: $\quad \phi(\alpha) : \forall x, \forall y((x < y \rightarrow y \in \alpha) \leftrightarrow x \in \alpha)$

Elements of $\overline{\mathbb{R}}_m$:

- $a : \mathbb{Q} \rightarrow \Omega$: $\quad \forall f, a_f = \{x \in \mathbb{Q}; f \in a(x)\}$ upper cut
- $\alpha : M \rightarrow \mathcal{P}(\mathbb{Q})^{op}$: $\forall f,\ \alpha(f)$ upper cut

Hence

- $\overline{\mathbb{R}}_m = \{\alpha : M \rightarrow \overline{\mathbb{R}}; \alpha \text{ monotone}\}$ (Reichman, 1983)
- $\overline{\mathbb{R}}_m = \{\mu : \mathcal{P}(\mathbb{N}) \rightarrow \overline{\mathbb{R}}; \mu \text{ monotone}, \mu(\emptyset) = -\infty\}$

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The bornological topos $\mathcal{B}$

Dense and closed $E \subseteq X^{\mathbb{N}}$

$\overline{E} = \Sigma(A)$, with $A = Ext(E) \subseteq X$ and the final bornology.

$$s \in \overline{E} \quad \text{if and only if} \quad \exists s_1, \ldots, s_n \in E, \;\; Im(s) \subseteq \bigcup_{1 \leq i \leq n} Im(s_i)$$

$E \subseteq X^{\mathbb{N}}$ is *dense* if $\overline{E} = X^{\mathbb{N}}$, and *closed* if $\overline{E} = E$.

- E is dense if and only if has a *finite covering*, that is,

$$\exists s_1, \ldots, s_n \in E, \; \mathbb{N} = \bigcup_{1 \leq i \leq n} Im(s_i)$$

- E is closed if and only if is *finitely determined*, that is,

$$(\exists s_1, \ldots, s_n \in E, Im(s) \subseteq \bigcup_{1 \leq i \leq n} Im(s_i)) \;\; \Rightarrow s \in E$$

...Finite coverings and sheaves

Case $X = \mathbb{N}$, ideals $I \subseteq M$.

- The dense ideals form a Grothendieck topology $\mathbb{J} \subseteq \Omega$ on M. The *bornological topos* is $\mathcal{B} = sh(M;\mathbb{J}) \hookrightarrow MSet$.
- Each $\Sigma(X)$ is a sheaf, $\Sigma : Born \to \mathcal{B}$.
- The sheafification on a set X is the M-set X_κ of all sequences $\mathbb{N} \to X$ with finite image.
- The subobject classifier of $\mathcal{B}$ is the M-subset $\Omega_b \subseteq \Omega$ of all closed ideals of M. Moreover $1+1 \cong 2_\kappa \cong \mathcal{P}(\mathbb{N})$.
- Rational number sheaf: $\mathbb{Q}_\kappa$.
- Real number sheaf: $\mathbb{R}_b = \ell^\infty$ (real bounded sequences)

$$C(\Omega_b) = \mathbb{R}_b^{\Omega_b} \cong \mathbb{R}_b \times \mathbb{R}_b \cong \mathbb{R}_b^{\mathcal{P}(\mathbb{N})}$$

$\mathcal{P}(\mathbb{N})$ and Ω_b

The inclusion $1 + 1 \hookrightarrow \Omega_b$ is a open morphism of locales
$(-)_\kappa \dashv Ext \dashv Cont : \mathcal{P}(\mathbb{N}) \hookrightarrow \Omega_b$

- $Cont(A) = \{f \in M; Im(f) \subseteq A\}$ $(content)$
- $Ext \circ (-)_\kappa = id = Ext \circ Cont$
- $Cont \circ Ext = \neg\neg$
- Frobenius identity: $(A \cap Ext(I))_\kappa = A_\kappa \cap I$

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Ω_b is isomorphic to the local of open set of the space $\beta\mathbb{N}$
Ω_b is the free regular compact local on the discrete local $\mathcal{P}(\mathbb{N})$.

Rational number object $\mathbb{Q}_\kappa$

Image finite sequences $s \in \mathbb{Q}_\kappa$

Display of $s : \mathbb{N} \longrightarrow \mathbb{Q}$, $Im(s) = \{x_1, \ldots, x_k\}$

$$1 \leq i \leq k \begin{cases} \mathbb{N} = \Sigma_i A_i, \;\; A_i = s^{-1}(x_i), \;\; s = \Sigma_i x_i e_{A_i} \\ I_i = \langle s = x_i \rangle = Cont(A_i) \in \Omega_b \\ I_s = \Sigma_i I_i = \{g \in M; s \circ g = cte\} \in \mathbb{J} \\ I_i = (g_i), \; Im(g_i) = A_i; \;\; \bigvee_i I_i = M \end{cases}$$

Definition of $\alpha : \mathbb{Q}_\kappa \to E$ by its constant level $\alpha_0 : \mathbb{Q} \to \Gamma(E)$

$$\exists! \alpha(s), \;\; \forall i, \;\; \alpha(s) \circ g_i = \alpha_0(x_i)$$

Parts of $\mathbb{Q}_\kappa$

Official: $\Omega_b^{\mathbb{Q}_\kappa} = \mathcal{B}(M \times \mathbb{Q}_\kappa, \Omega_b)$

$$\bar{a} : M \times \mathbb{Q}_\kappa \to \Omega_b, \ \ (\bar{a} \circ f)(g, s) = \bar{a}(f \circ g, s)$$

Free sheaf: $\hat{a} : M \times \mathbb{Q} \to \Omega_b$

Practical: $\Omega_b^{\mathbb{Q}_\kappa} \cong \Omega_b^{\mathbb{Q}} = Set(\mathbb{Q}, \Omega_b)$

$$a : \mathbb{Q} \to \Omega_b, \ \ (a \circ f)(x) = \langle f \in a(x) \rangle$$

From a to $\bar{a}$:

- $\hat{a}(f, x) = (a \circ f)(x)$
- $\bar{a}(f, s) = \bigvee_i (I_i \cap \langle f \in a(x_i) \rangle)$

Set theory of $\mathbb{Q}_\kappa$

$$(=) \hookrightarrow \mathbb{Q}_\kappa \times \mathbb{Q}_\kappa \to \Omega_b, \quad \langle s = t \rangle = \bigvee_{x_i = y_j} (I_i \cap J_j)$$

Free sheaf: $\mathbb{Q} \times \mathbb{Q} \to \{\emptyset, M\} \hookrightarrow \Omega_b$

$$at : \mathbb{Q}_\kappa \to \Omega_b^{\mathbb{Q}}, \quad at(s)(x) = \langle s = x \rangle = \begin{cases} I_i, & x = x_i,\ 1 \le i \le k \\ \emptyset, & x \notin Im(s) \end{cases}$$

Free sheaf: $at_0 : \mathbb{Q} \to \mathcal{P}(\mathbb{Q}), \quad at_0(x) = \{x\}$

$$ev : \Omega_b^{\mathbb{Q}} \times \mathbb{Q}_\kappa \to \Omega_b, \quad a(s) : \quad I_i \cap a(s) = I_i \cap a(x_i),\ 1 \le i \le k$$

Free sheaf: $ev : \Omega_b^{\mathbb{Q}} \times \mathbb{Q} \to \Omega_b, \quad ev(a, x) = a(x)$

$$s \in a \quad \Leftrightarrow \quad I_i \subseteq a(x_i),\ 1 \le i \le k \quad \Leftrightarrow \quad at(s) \subseteq a$$

$$s < s' \quad \Leftrightarrow \quad \forall i, j\ (I_i \cap I'_j \neq \emptyset \ \Rightarrow\ x_i < x'_j)$$

Variations of $\Omega_b^{\mathbb{Q}}$

Recall:

- $a : \mathbb{Q} \to \Omega \subseteq \mathcal{P}(M)$, $\quad a(x) = \{f \in M; x \in \alpha(f)\}$
- $\alpha : M \to \mathcal{P}(\mathbb{N})^{op}$ monotone, $\quad \alpha(f) = \{x \in \mathbb{Q}; f \in a(x)\}$
- $\mu : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{Q})^{op}$ monotone, $\mu(A) = \alpha(f),\ A = Im(f),\ \mu(\emptyset) = \mathbb{Q}$

Now are equivalent:

- a factorizes throught Ω_b
- $(g_1, \dots, g_n) \in \mathbb{J} \ \Rightarrow\ \alpha(f) = \bigcap_i \alpha(f \circ g_i)$
- $A = \bigcup_i A_i \ \Rightarrow\ \mu(A) = \bigcap_i \mu(A_i),\ (1 \leq i \leq n)$

Set theory: $s \in \mu \ \Leftrightarrow\ x_i \in \mu(A_i),\ 1 \leq i \leq k$

Extended real number object $\overline{\mathbb{R}}_b$ in $\mathcal{B}$

$\overline{\mathbb{R}}_b \subseteq \Omega_b^{\mathbb{Q}}$: $\phi(\alpha)$: $\forall x, \forall y((x < y \to y \in \alpha) \leftrightarrow x \in \alpha)$

Recall $\overline{\mathbb{R}}_m$ in *MSet*. Elements: $\alpha : M \to \mathcal{P}(\mathbb{Q})^{op}$:

- $\forall f$, $\alpha(f)$ is an upper cut, and α monotone
- $\overline{\mathbb{R}}_m = \{\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}; \mu \text{ monotone}, \mu(\emptyset) = -\infty\}$

Now $\overline{\mathbb{R}}_b$ in $\mathcal{B}$. Elements: $\alpha : M \to \mathcal{P}(\mathbb{Q})^{op}$:

- $\forall f$, $\alpha(f)$ is an upper cut, and α factorizes through Ω_b
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$\overline{\mathbb{R}}_b$ is non-separated ($\Gamma(\overline{\mathbb{R}}_b) \cong \overline{\mathbb{R}}$)
$\overline{\mathbb{R}}_b$ is an internal local.

...

Relating $\mathbb{R}_b$ and $\overline{\mathbb{R}}_b$

- $\mathbb{R}_b \hookrightarrow \overline{\mathbb{R}}_b, \;\; s(A) = sup_{n \in A} s(n)$
- $|-| : \mathbb{R}_b \to \overline{\mathbb{R}}_b^+, \;\; |s|(A) = sup_{n \in A} |s(x)|$
- $\overline{\mathbb{R}}_b^+ = \{\mu : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}^+; \mu$ preserves finite $\vee\}$ (semiring)

$\overline{\mathbb{R}}_b^+$ has the properties we need to study internal normed linear spaces with norms valued on $\overline{\mathbb{R}}_b^+$

To be continued ... CT200?