Topological spaces, categorically

Dirk Hofmann
dirk@mat.ua.pt

University of Aveiro

CT 2007

The talk is based on joint work with M.M. Clementino and W. Tholen.

Motivation

"The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that *fundamental* structures are themselves categories."

F. William Lawvere.
Metric spaces, generalized logic, and closed categories.
Rend. Sem. Mat. Fis. Milano, 43:135–166 (1974), 1973.
Also in: *Repr. Theory Appl. Categ.* 1:1–37, 2002.

Examples

Metric spaces, $(\mathsf{P}_+ = [0, \infty]^{\mathrm{op}}, +, 0)$

X with $d : X \times X \longrightarrow \mathsf{P}_+$ such that

$$0 \geq d(x,x), \qquad d(x,y) + d(y,z) \geq d(x,z).$$

Categories, $(\mathrm{Set}, \times, 1)$

X with $\mathrm{hom} : X \times X \longrightarrow \mathrm{Set}$ such that

$$1 \longrightarrow \mathrm{hom}(x,x), \qquad \mathrm{hom}(x,y) \times \mathrm{hom}(y,z) \longrightarrow \mathrm{hom}(x,z)$$

and . . . (commutative diagrams in Set).

Examples

Metric spaces, $(\mathsf{P}_+ = [0, \infty]^{\mathrm{op}}, +, 0)$

X with $d : X \times X \longrightarrow \mathsf{P}_+$ such that

$$0 \geq d(x,x), \qquad d(x,y) + d(y,z) \geq d(x,z).$$

Categories, $(\mathrm{Set}, \times, 1)$

X with $\hom : X \times X \longrightarrow \mathrm{Set}$ such that

$$1 \longrightarrow \hom(x,x), \qquad \hom(x,y) \times \hom(y,z) \longrightarrow \hom(x,z)$$

and . . . (commutative diagrams in Set).

Ordered sets, $(2 = \{\mathrm{false}, \mathrm{true}\}, \&, \mathrm{true})$

X with $\leq : X \times X \longrightarrow 2$ such that

$$\mathrm{true} \models (x \leq x), \qquad (x \leq y \,\&\, y \leq z) \models x \leq z.$$

The ordered category V-Rel

Quantale

$\mathsf{V} = (\mathsf{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

The ordered category V-Rel

Quantale

$\mathsf{V} = (\mathsf{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

V-Rel

- Objects: sets X, Y,...

The ordered category V-Rel

Quantale

$\mathsf{V} = (\mathsf{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

V-Rel

- Objects: sets X, Y,...
- Morphisms: V-relations $r : X \times Y \longrightarrow \mathsf{V}$; we write $r : X \relbar\!\!\!\mapsto Y$

The ordered category V-Rel

Quantale

V $= (\mathrm{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

V-Rel

- Objects: sets X, Y,...
- Morphisms: V-relations $r : X \times Y \longrightarrow \mathrm{V}$; we write $r : X \relbar\joinrel\mapsto Y$
- Composition: (with $s : Y \relbar\joinrel\mapsto Z$)

$$s \cdot r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$$

The ordered category V-Rel

Quantale

$\mathsf{V} = (\mathsf{V}, \otimes, k)$ commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

V-Rel

- Objects: sets X, Y,...
- Morphisms: V-relations $r : X \times Y \longrightarrow \mathsf{V}$; we write $r : X \relbar\!\!\!\mapsto Y$
- Composition: (with $s : Y \relbar\!\!\!\mapsto Z$)

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

- Involution: $r^\circ : Y \relbar\!\!\!\mapsto X$ where $r^\circ(y, x) = r(x, y)$ for $r : X \relbar\!\!\!\mapsto Y$.

The ordered category V-Rel

Quantale

V $=$ (V, $\otimes$, k) commutative unital quantale with $u \otimes \_ \dashv \hom(u, \_)$.

V-Rel

- Objects: sets X, Y, ...
- Morphisms: V-relations $r : X \times Y \longrightarrow \mathsf{V}$; we write $r : X \relbar\!\!\mapsto Y$
- Composition: (with $s : Y \relbar\!\!\mapsto Z$)

$$s \cdot r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$$

- Involution: $r^\circ : Y \relbar\!\!\mapsto X$ where $r^\circ(y,x) = r(x,y)$ for $r : X \relbar\!\!\mapsto Y$.
- For each Set-map f: $f \dashv f^\circ$.

V-Cat

V-categories

A V-category is a pair $(X, a : X \to\!\!\!\!\!\!+\;\; X)$ such that

$$k \leq a(x,x) \qquad\qquad a(x,y) \otimes a(y,z) \leq a(x,z)$$

respectively

$$\mathrm{id}_X \leq a \qquad\qquad a \cdot a \leq a$$

V-Cat

V-categories

A V-category is a pair $(X, a : X \relbar\joinrel\mapsto X)$ such that

$$k \leq a(x,x) \qquad\qquad a(x,y) \otimes a(y,z) \leq a(x,z)$$

respectively

$$\mathrm{id}_X \leq a \qquad\qquad a \cdot a \leq a$$

V-functors

A V-functor $f : (X, a) \longrightarrow (Y, b)$ is a Set-map such that

$$a(x,x') \leq b(f(x), f(x')) \qquad \text{respectively} \qquad f \cdot a \leq b \cdot f.$$

M. Barr 1970

Topological spaces $\quad \mathsf{2} = (2, \&, \mathrm{true}), \quad \mathbb{U} = (U, e, m)$

X with $\longrightarrow : UX \relbar\joinrel\mapstochar\rightarrow X$ such that

$$\mathrm{true} \models (\dot{x} \longrightarrow x), \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \,\&\, \mathfrak{x} \longrightarrow x) \models m_X(\mathfrak{X}) \longrightarrow x.$$

Here $\longrightarrow : UX \relbar\joinrel\mapstochar\rightarrow X$ is naturally extended to $\longrightarrow : UUX \relbar\joinrel\mapstochar\rightarrow UX$.

M. Barr 1970

Topological spaces $\quad 2 = (2, \&, \mathrm{true}), \quad \mathbb{U} = (U, e, m)$

X with $\longrightarrow : UX \longrightarrow\!\!\!\!\!\mapsto X$ such that

$$\mathrm{true} \models (\dot{x} \longrightarrow x), \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \,\&\, \mathfrak{x} \longrightarrow x) \models m_X(\mathfrak{X}) \longrightarrow x.$$

Here $\longrightarrow : UX \longrightarrow\!\!\!\!\!\mapsto X$ is naturally extended to $\longrightarrow : UUX \longrightarrow\!\!\!\!\!\mapsto UX$.

In fact, $U : \mathrm{Set} \longrightarrow \mathrm{Set}$ can be extended to a functor
$U : \mathrm{Rel} \longrightarrow \mathrm{Rel}$ such that e and m become oplax.

Some facts about V-categories

1. V-Cat is a monoidal closed category.

Some facts about V-categories

1. V-Cat is a monoidal closed category.
2. $V = (V, \hom)$ is a (complete) V-category.

Some facts about V-categories

1. V-Cat is a monoidal closed category.
2. $\mathsf{V} = (\mathsf{V}, \hom)$ is a (complete) V-category.
3. $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$ is a V-module iff $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ is a V-functor.

Some facts about V-categories

1. V-Cat is a monoidal closed category.
2. $\mathsf{V} = (\mathsf{V}, \hom)$ is a (complete) V-category.
3. $\varphi : X \multimap\!\!\rightarrow Y$ is a V-module iff $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ is a V-functor.
4. In particular $a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$ is a V-functor. Its mate $y = \ulcorner a \urcorner : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$ is fully faithful. More general, we have
$$[y(x), \varphi] = \varphi(x).$$
5. ...

Topological theory

Definition

A topological theory $\mathcal{T}$ is a triple $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of

a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and

a map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$

such that

Topological theory

Definition

A topological theory $\mathcal{T}$ is a triple $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consisting of

a monad $\mathbb{T} = (T, e, m)$, a quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$ and

a map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$

such that

$(\mathrm{M_e})$ $\mathrm{id}_{\mathsf{V}} \leq \xi \cdot e_{\mathsf{V}}$, $\qquad$ $(\mathrm{M_m})$ $\xi \cdot T\xi \leq \xi \cdot m_{\mathsf{V}}$,

$(\mathrm{Q}_\otimes)$
$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\otimes)} & T\mathsf{V} \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\otimes]{} & \mathsf{V}, \end{array}$$

$(\mathrm{Q_k})$
$$\begin{array}{ccc} T1 & \xrightarrow{Tk} & T\mathsf{V} \\ {\scriptstyle !}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[k]{} & \mathsf{V}, \end{array}$$

$(\mathrm{Q}_\vee)$ $(\xi_X)_X : P_{\mathsf{v}} \longrightarrow P_{\mathsf{v}} T$ is a natural transformation.

Examples

- $\mathcal{I}_{\mathsf{V}} = (\mathbb{I}, \mathsf{V}, \mathrm{id}_{\mathsf{V}})$ is a strict topological theory.

Examples

- $\mathcal{I}_{\mathsf{V}} = (\mathbb{I}, \mathsf{V}, \mathsf{id}_{\mathsf{V}})$ is a strict topological theory.
- $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory.

Examples

- $\mathcal{I}_{\mathsf{V}} = (\mathbb{I}, \mathsf{V}, \mathrm{id}_{\mathsf{V}})$ is a strict topological theory.
- $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory.
- $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ is a strict topological theory, where

$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \quad \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid \mathfrak{x} \in T([0, v])\}.$$

Examples

- $\mathcal{I}_V = (\mathbb{I}, V, \mathrm{id}_V)$ is a strict topological theory.
- $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory.
- $\mathcal{U}_{P_+} = (\mathbb{U}, P_+, \xi_{P_+})$ is a strict topological theory, where

$$\xi_{P_+} : UP_+ \longrightarrow P_+, \quad \mathfrak{x} \longmapsto \inf\{v \in P_+ \mid \mathfrak{x} \in T([0, v])\}.$$

- $\mathcal{T}_V = (\mathbb{T}, V, \xi_V)$ where T satisfies (BC), V is (ccd) and

$$\xi_V : TV \longrightarrow V, \quad \mathfrak{x} \longmapsto \bigvee\{v \in V \mid \mathfrak{x} \in T(\uparrow v)\}.$$

Examples

- $\mathcal{I}_{\mathsf{V}} = (\mathbb{I}, \mathsf{V}, \mathrm{id}_{\mathsf{V}})$ is a strict topological theory.
- $\mathcal{U}_{2} = (\mathbb{U}, 2, \xi_{2})$ is a strict topological theory.
- $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ is a strict topological theory, where

$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \quad \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid \mathfrak{x} \in T([0, v])\}.$$

- $\mathcal{T}_{\mathsf{V}} = (\mathbb{T}, \mathsf{V}, \xi_{\mathsf{V}})$ where T satisfies (BC), V is (ccd) and

$$\xi_{\mathsf{V}} : T\mathsf{V} \longrightarrow \mathsf{V}, \quad \mathfrak{x} \longmapsto \bigvee\{v \in \mathsf{V} \mid \mathfrak{x} \in T(\uparrow v)\}.$$

- $\mathcal{L}_{\mathsf{V}}^{\otimes} = (\mathbb{L}, \mathsf{V}, \xi_{\otimes})$ is a strict topological theory where

$$\xi_{\otimes} : L\mathsf{V} \longrightarrow \mathsf{V}, \quad (v_1, \dots, v_n) \longmapsto v_1 \otimes \dots \otimes v_n.$$

Extending T : Set $\longrightarrow$ Set to V-Rel

We define T_ξ : V-Rel $\longrightarrow$ V-Rel as follows:

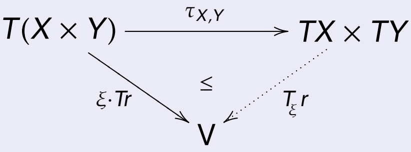
Extending $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ to V-Rel

We define $T_\xi : \text{V-Rel} \longrightarrow \text{V-Rel}$ as follows:

Given $r : X \times Y \longrightarrow \mathsf{V}$, we put

$$
\begin{aligned}
T_\xi r : TX \times TY &\longrightarrow \mathsf{V} \\
(\mathfrak{x}, \mathfrak{y}) &\longmapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \,\middle|\, \mathfrak{w} \in T(X \times Y), \mathfrak{w} \longmapsto \mathfrak{x}, \mathfrak{y} \right\},
\end{aligned}
$$

that is,



Properties of T_ξ

Theorem

The following statements hold.

1. *For each* V*-relation* $r : X \relbar\joinrel\mapsto Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$.

Properties of T_ξ

Theorem

The following statements hold.

1. *For each* V*-relation* $r : X \relbar\!\!\!\mapsto Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$.
2. *For each function* $f : X \longrightarrow Y$, $Tf \leq T_\xi f$ *and* $Tf^\circ \leq T_\xi f^\circ$.

Properties of T_ξ

Theorem

The following statements hold.

1. *For each* V*-relation* $r : X \relbar\!\!\!\mapsto Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$.
2. *For each function* $f : X \longrightarrow Y$, $Tf \le T_\xi f$ *and* $Tf^\circ \le T_\xi f^\circ$.
3. $T_\xi s \cdot T_\xi r \le T_\xi(s \cdot r)$ *provided that* T *satisfies (BC), and* $T_\xi s \cdot T_\xi r \ge T_\xi(s \cdot r)$ *provided that* $(Q^{=}_{\otimes})$ *holds.*

Properties of T_ξ

Theorem

The following statements hold.

1. *For each* V*-relation* $r : X \nrightarrow Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$.
2. *For each function* $f : X \longrightarrow Y$, $Tf \leq T_\xi f$ *and* $Tf^\circ \leq T_\xi f^\circ$.
3. $T_\xi s \cdot T_\xi r \leq T_\xi(s \cdot r)$ *provided that* T *satisfies (BC), and* $T_\xi s \cdot T_\xi r \geq T_\xi(s \cdot r)$ *provided that* $(Q^{=}_{\otimes})$ *holds.*
4. *The natural transformations* e *and* m *become op-lax, that is, for every* V*-relation* $r : X \nrightarrow Y$ *we have the inequalities:*

$$\begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\downarrow\!\!\!\!\!+ & \leq & \downarrow\!\!\!\!\!+ {\scriptstyle T_\xi r} \\ Y & \xrightarrow[e_Y]{} & TY \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle T_\xi T_\xi r}\downarrow\!\!\!\!\!+ & \leq & \downarrow\!\!\!\!\!+ {\scriptstyle T_\xi r} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array}$$

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets $X, Y, \ldots$

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets $X, Y, \ldots$
- morphisms: $\mathcal{T}$-relations $a : X \rightharpoondown\!\!\!\!\!\!\!\!\mapsto Y$, i.e. V-relations $a : TX \rightarrow\!\!\!\!\!\!\!\!\mapsto Y$.

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets X, Y, ...
- morphisms: $\mathcal{T}$-relations $a : X \rightharpoondown\!\!\!\!+ Y$, i.e. V-relations $a : TX \nrightarrow Y$.
- The Kleisli convolution of $a : X \rightharpoondown\!\!\!\!+ Y$ and $b : Y \rightharpoondown\!\!\!\!+ Z$ is defined as

$$b \circ a = b \cdot T_{\xi} a \cdot m_X^{\circ}.$$

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets X, Y, ...
- morphisms: $\mathcal{T}$-relations $a : X \rightharpoondown\!\!\!\!\!+ Y$, i.e. V-relations $a : TX \to\!\!\!\!\!+ Y$.
- The Kleisli convolution of $a : X \rightharpoondown\!\!\!\!\!+ Y$ and $b : Y \rightharpoondown\!\!\!\!\!+ Z$ is defined as

$$b \circ a = b \cdot T_{\xi} a \cdot m_X^{\circ}.$$

Some properties

We have

- $a \circ e_X^{\circ} \geq a$ and $e_Y^{\circ} \circ a \geq a$.

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets $X, Y, \ldots$
- morphisms: $\mathcal{T}$-relations $a : X \rightharpoondown\!\!\!\!\!\!\!\!\!\!\;\;\;\mid\; Y$, i.e. V-relations $a : TX \rightarrow\!\!\!\!\!\mapsto Y$.
- The Kleisli convolution of $a : X \rightharpoondown\!\!\!\!\!\!\!\!\!\!\;\;\;\mid\; Y$ and $b : Y \rightharpoondown\!\!\!\!\!\!\!\!\!\!\;\;\;\mid\; Z$ is defined as

$$b \circ a = b \cdot T_{\xi} a \cdot m_X^{\circ}.$$

Some properties

We have

- $a \circ e_X^{\circ} \geq a$ and $e_Y^{\circ} \circ a \geq a$.
- $a \circ (b \circ c) \geq a \circ b \circ c \leq (a \circ b) \circ c$.

Kleisli convolution

$\mathcal{T}$-Rel

- objects: sets X, Y, ...
- morphisms: $\mathcal{T}$-relations $a : X \relbar\joinrel\rightharpoonup Y$, i.e. V-relations $a : TX \relbar\joinrel\mapsto Y$.
- The Kleisli convolution of $a : X \relbar\joinrel\rightharpoonup Y$ and $b : Y \relbar\joinrel\rightharpoonup Z$ is defined as

$$b \circ a = b \cdot T_{\xi} a \cdot m_X^\circ.$$

Some properties

We have

- $a \circ e_X^\circ \geq a$ and $e_Y^\circ \circ a \geq a$.
- $a \circ (b \circ c) \geq a \circ b \circ c \leq (a \circ b) \circ c$.
- If $\mathcal{T}$ is a strict theory, then Kleisli convolution is associative.

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \relbar\!\mapsto Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

We consider now

$$(\_)_\# : \mathsf{V}\text{-}\mathsf{Rel} \longrightarrow \mathcal{T}\text{-}\mathsf{Rel}, \quad r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y \longmapsto r_\# = e_Y \cdot T_\xi r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$$

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

We consider now

$$(\_)_\# : \text{V-Rel} \longrightarrow \mathcal{T}\text{-Rel}, \quad r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y \longmapsto r_\# = e_Y \cdot T_\xi r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$$

We have

- $(1_Y)_\# \circ a = e_Y^\circ \circ a$ and $a \circ (1_X)_\# = a \circ e_X^\circ$.

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \relbar\mapsto Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

We consider now

$$(\_)_\# : \text{V-Rel} \longrightarrow \mathcal{T}\text{-Rel}, \quad r : X \relbar\mapsto Y \longmapsto r_\# = e_Y \cdot T_\xi r : X \relbar\mapsto Y$$

We have

- $(1_Y)_\# \circ a = e_Y^\circ \circ a$ and $a \circ (1_X)_\# = a \circ e_X^\circ$.
- $r_\#$ is unitary.

V-Rel vs. $\mathcal{T}$-Rel

We call $a : X \nrightarrow Y$ unitary if $e_Y^\circ \circ a = a$ and $a \circ e_X^\circ = a$.

We consider now

$$(\_)_\# : \mathsf{V}\text{-}\mathrm{Rel} \longrightarrow \mathcal{T}\text{-}\mathrm{Rel}, \quad r : X \nrightarrow Y \longmapsto r_\# = e_Y \cdot T_\xi r : X \nrightarrow Y$$

We have

- $(1_Y)_\# \circ a = e_Y^\circ \circ a$ and $a \circ (1_X)_\# = a \circ e_X^\circ$.
- $r_\#$ is unitary.
- T satisfies (BC) $\quad \Rightarrow \quad s_\# \circ r_\# \leq (s \cdot r)_\#$.
- $(Q_\otimes^=) \quad \Rightarrow \quad s_\# \circ r_\# \geq (s \cdot r)_\#$.

$\mathcal{T}$-Cat

$\mathcal{T}$-category

A $\mathcal{T}$–category is a pair $(X, a : TX \longrightarrow\!\!\!\!\!\!\mapsto X)$ such that

$$k \le a(e_X(x), x), \quad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x) \quad \text{respectively}$$

$$\mathrm{id}_X \le a \cdot e_X, \quad a \cdot T_\xi a \le a \cdot m_X \quad \text{respectively}$$

$$e_X^\circ \le a, \quad a \circ a \le a.$$

$\mathcal{T}$-Cat

$\mathcal{T}$-category

A $\mathcal{T}$–category is a pair $(X, a : TX \longrightarrow\!\!\!\!\!\mapsto X)$ such that

$$k \leq a(e_X(x), x), \quad T_{\xi} a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x) \quad \text{respectively}$$

$$\mathrm{id}_X \leq a \cdot e_X, \quad a \cdot T_{\xi} a \leq a \cdot m_X \quad \text{respectively}$$

$$e_X^{\circ} \leq a, \quad a \circ a \leq a.$$

$\mathcal{T}$-functor

A map $f : (X, a) \longrightarrow (Y, b)$ is a $\mathcal{T}$-functor if

$$a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x)) \quad \text{respectively} \quad f \cdot a \leq b \cdot Tf.$$

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.
- In particular, $\mathcal{I}_2$-Cat $\cong$ Ord and $\mathcal{I}_{P_+}$-Cat $\cong$ Met.

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.
- In particular, $\mathcal{I}_2$-Cat $\cong$ Ord and $\mathcal{I}_{P_+}$-Cat $\cong$ Met.
- $\mathcal{U}_2$-Cat $\cong$ Top.

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.
- In particular, $\mathcal{I}_2$-Cat $\cong$ Ord and $\mathcal{I}_{P_+}$-Cat $\cong$ Met.
- $\mathcal{U}_2$-Cat $\cong$ Top.
- $\mathcal{U}_{P_+}$-Cat $\cong$ App.

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.
- In particular, $\mathcal{I}_2$-Cat $\cong$ Ord and $\mathcal{I}_{P_+}$-Cat $\cong$ Met.
- $\mathcal{U}_2$-Cat $\cong$ Top.
- $\mathcal{U}_{P_+}$-Cat $\cong$ App.
- $\mathcal{L}_V^{\otimes}$-Cat $\cong$ V-MultiCat.

Examples

- For each quantale V, $\mathcal{I}_V$-Cat $\cong$ V-Cat.
- In particular, $\mathcal{I}_2$-Cat $\cong$ Ord and $\mathcal{I}_{P_+}$-Cat $\cong$ Met.
- $\mathcal{U}_2$-Cat $\cong$ Top.
- $\mathcal{U}_{P_+}$-Cat $\cong$ App.
- $\mathcal{L}_V^{\otimes}$-Cat $\cong$ V-MultiCat.

From now on we consider a strict theory $\mathcal{T} = (\mathbb{T}, V, \xi)$.

Some functors

We have an embedding $\mathsf{Set}^{\mathbb{T}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and put $|X| = (TX, m_X)$.

Some functors

We have an embedding $\mathsf{Set}^{\mathbb{T}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and put $|X| = (TX, m_X)$.

We have $(\_)_\# \dashv \mathsf{S}$ where

$$\begin{aligned} \mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} &\longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \\ (X, a) &\longmapsto (X, a \cdot e_X) \end{aligned} \qquad \begin{aligned} (\_)_\# : \mathsf{V}\text{-}\mathsf{Cat} &\longrightarrow \mathcal{T}\text{-}\mathsf{Cat}. \\ X = (X, r) &\longmapsto X_\# = (X, r_\#) \end{aligned}$$

Some functors

We have an embedding $\mathsf{Set}^{\mathbb{T}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and put $|X| = (TX, m_X)$.

We have $(\_)_\# \dashv \mathsf{S}$ where

$$\mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \qquad (\_)_\# : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}.$$

$$(X, a) \longmapsto (X, a \cdot e_X) \qquad X = (X, r) \longmapsto X_\# = (X, r_\#)$$

T_ξ induces an endofunctor

$$T_\xi : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \qquad (X, r) \longmapsto (TX, T_\xi r)$$

Some functors

We have an embedding $\mathsf{Set}^{\mathbb{T}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and put $|X| = (TX, m_X)$.

We have $(\_)_\# \dashv \mathsf{S}$ where

$$\begin{aligned} \mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} &\longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, & (\_)_\# : \mathsf{V}\text{-}\mathsf{Cat} &\longrightarrow \mathcal{T}\text{-}\mathsf{Cat}. \\ (X, a) &\longmapsto (X, a \cdot e_X) & X = (X, r) &\longmapsto X_\# = (X, r_\#) \end{aligned}$$

T_ξ induces an endofunctor

$$T_\xi : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}, \qquad (X, r) \longmapsto (TX, T_\xi r)$$

and we have

$$\begin{array}{ccccc} & & \mathcal{T}\text{-}\mathsf{Cat} & & \\ & {\scriptstyle (\_)_\#}\nearrow & & \searrow{\scriptstyle \mathsf{M}} & \\ \mathsf{V}\text{-}\mathsf{Cat} & & \xrightarrow[T_\xi]{\qquad\qquad} & & \mathsf{V}\text{-}\mathsf{Cat} \end{array}$$

where $\mathsf{M} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat},\ (X, a) \longmapsto (TX, T_\xi a \cdot m_X^\circ)$.

The $\mathcal{T}$-category V

We define

$$\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V},\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is a $\mathcal{T}$-category.

The $\mathcal{T}$-category V

We define

$$\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V},\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is a $\mathcal{T}$-category.

Some maps

1. $\bigwedge : \mathsf{V}^I \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor.

The $\mathcal{T}$-category V

We define

$$\hom_\xi : TV \times V \longrightarrow V,\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $V = (V, \hom_\xi)$ is a $\mathcal{T}$-category.

Some maps

1. $\bigwedge : V^I \longrightarrow V$ is a $\mathcal{T}$-functor.
2. $\hom(v, \_) : V \longrightarrow V$ is a $\mathcal{T}$-functor for each $v \in V$ which satisfies $\xi \cdot Tv \geq v \cdot !$.

The $\mathcal{T}$-category V

We define

$$\hom_\xi : TV \times V \longrightarrow V, \ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Then $V = (V, \hom_\xi)$ is a $\mathcal{T}$-category.

Some maps

1. $\bigwedge : V^I \longrightarrow V$ is a $\mathcal{T}$-functor.
2. $\hom(v, \_) : V \longrightarrow V$ is a $\mathcal{T}$-functor for each $v \in V$ which satisfies $\xi \cdot Tv \geq v \cdot !$.
3. $v \otimes \_ : V \longrightarrow V$ is a $\mathcal{T}$-functor for each $v \in V$ which satisfies $\xi \cdot Tv \leq v \cdot !$.

Compatible monoidal structures on V

We assume that a monoidal structure $(\mathrm{V}, \oplus, I)$ on V is given such that

1. $(u_1 \oplus v_1) \otimes (u_2 \oplus v_2) \leq (u_1 \otimes u_2) \oplus (v_1 \otimes v_2)$,
2. $I \otimes I \leq I$ and $k \leq k \oplus k$,

Compatible monoidal structures on V

We assume that a monoidal structure $(\mathsf{V}, \oplus, I)$ on V is given such that

1. $(u_1 \oplus v_1) \otimes (u_2 \oplus v_2) \leq (u_1 \otimes u_2) \oplus (v_1 \otimes v_2)$,
2. $I \otimes I \leq I$ and $k \leq k \oplus k$,
3. $$\begin{array}{ccc} T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T(\oplus)} & T\mathsf{V} \\ {\scriptstyle \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle} \downarrow & \geq & \downarrow {\scriptstyle \xi} \\ \mathsf{V} \times \mathsf{V} & \xrightarrow[\oplus]{} & \mathsf{V}, \end{array} \quad \text{and} \quad \begin{array}{ccc} T1 & \xrightarrow{TI} & T\mathsf{V} \\ {\scriptstyle !} \downarrow & \geq & \downarrow {\scriptstyle \xi} \\ 1 & \xrightarrow[I]{} & \mathsf{V}. \end{array}$$

Compatible monoidal structures on V

We assume that a monoidal structure $(\mathsf{V}, \oplus, I)$ on V is given such that

1. $(u_1 \oplus v_1) \otimes (u_2 \oplus v_2) \leq (u_1 \otimes u_2) \oplus (v_1 \otimes v_2)$,
2. $I \otimes I \leq I$ and $k \leq k \oplus k$,
3. $$\begin{array}{ccc} T(\mathsf{V} \times \mathsf{V}) & \xrightarrow{T(\oplus)} & T\mathsf{V} \\ {\scriptstyle \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle} \downarrow & \geq & \downarrow {\scriptstyle \xi} \\ \mathsf{V} \times \mathsf{V} & \xrightarrow[\oplus]{} & \mathsf{V}, \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T1 & \xrightarrow{TI} & T\mathsf{V} \\ {\scriptstyle !} \downarrow & \geq & \downarrow {\scriptstyle \xi} \\ 1 & \xrightarrow[I]{} & \mathsf{V}. \end{array}$$

Examples

- $\oplus = \otimes$ (since $\mathcal{T}$ is strict).
- $\oplus = \wedge$.

Monoidal structures on V-Rel

Extending $\oplus$ to V-Rel

- For sets X and Y we put $X \oplus Y = X \times Y$.
- For V-relations $r : X \relbar\joinrel\mapsto X'$ and $s : Y \relbar\joinrel\mapsto Y'$ we define $r \oplus s : X \times Y \relbar\joinrel\mapsto X' \times Y'$ by

$$r \oplus s((x, y), (x', y')) = r(x, x') \oplus s(y, y').$$

Then $\oplus$: V-Rel $\times$ V-Rel $\longrightarrow$ V-Rel is a lax functor, is associative and with $I : 1 \relbar\joinrel\mapsto 1$ as neutral element.

Monoidal structures on V-Rel

Extending $\oplus$ to V-Rel

- For sets X and Y we put $X \oplus Y = X \times Y$.
- For V-relations $r : X \relbar\!\!\!\mapsto X'$ and $s : Y \relbar\!\!\!\mapsto Y'$ we define $r \oplus s : X \times Y \relbar\!\!\!\mapsto X' \times Y'$ by

$$r \oplus s((x, y), (x', y')) = r(x, x') \oplus s(y, y').$$

Then $\oplus : \text{V-Rel} \times \text{V-Rel} \longrightarrow \text{V-Rel}$ is a lax functor, is associative and with $I : 1 \relbar\!\!\!\mapsto 1$ as neutral element.

Of course, we obtain a monoidal structure on V-Cat where $(X, a) \oplus (Y, b) = (X \times Y, a \oplus b)$ with neutral element $E = (1, I)$.

I. Moerdijk, 1999

Hopf monad

A Hopf monad on a monoidal category E is a monad $\mathbb{T} = (T, e, m)$ on E equipped with a natural transformation

$$\tau : T(\_ \otimes \_) \longrightarrow T(\_) \otimes T(\_)$$

and a map $\theta : T(N) \longrightarrow N$ such that ...

I. Moerdijk, 1999

Hopf monad

A Hopf monad on a monoidal category E is a monad $\mathbb{T} = (T, e, m)$ on E equipped with a natural transformation

$$\tau : T(\_ \otimes \_) \longrightarrow T(\_) \otimes T(\_)$$

and a map $\theta : T(N) \longrightarrow N$ such that ...

Theorem

There is a bijective correspondence between such structures τ, θ on $\mathbb{T}$ and liftings of the monoidal structure on E *to* $\mathrm{E}^{\mathbb{T}}$.

Here:

$$(X, \alpha) \otimes (Y, \beta) = (X \otimes Y, (\alpha \otimes \beta) \cdot \tau_{X,Y}).$$

Lax Hopf monad

With $\tau_{X,Y} : T(X \times Y) \longrightarrow TX \times TY$ and $! : T1 \longrightarrow 1$, in our situation we have

$$\begin{array}{ccc} T(X \oplus Y) & \xrightarrow{\tau_{X,Y}} & TX \oplus TY \\ {\scriptstyle T_\xi(r \oplus s)} \big\downarrow \!\!\!\!\!\! + & \leq & + \!\!\!\!\!\! \big\downarrow {\scriptstyle T_\xi r \oplus T_\xi s} \\ T(X' \oplus Y') & \xrightarrow[\tau_{X',Y'}]{} & TX' \oplus TY' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T1 & \xrightarrow{!} & 1 \\ {\scriptstyle T_\xi l} \big\downarrow \!\!\!\!\!\! + & \leq & + \!\!\!\!\!\! \big\downarrow {\scriptstyle l} \\ T1 & \xrightarrow[!]{} & 1 \end{array}$$

making (T_ξ, e, m) a lax Hopf monad on V-Rel.

Extending $\oplus$ to $\mathcal{T}$-Rel...

Let $r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y'$ be $\mathcal{T}$-relations. We put $X \boxplus Y = X \times Y$ and define $r \boxplus s : X \times Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup X' \times Y'$ as

$$r \boxplus s = (r \oplus s) \cdot \tau_{X,Y}.$$

and $l_! : 1 \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ as the composite $T1 \xrightarrow{!} 1 \xrightarrow{l} 1$.

Extending $\oplus$ to $\mathcal{T}$-Rel...

Let $r : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y'$ be $\mathcal{T}$-relations. We put $X \boxplus Y = X \times Y$ and define $r \boxplus s : X \times Y \relbar\joinrel\mapstochar\joinrel\rightharpoonup X' \times Y'$ as

$$r \boxplus s = (r \oplus s) \cdot \tau_{X,Y}.$$

and $l_! : 1 \relbar\joinrel\mapstochar\joinrel\rightharpoonup 1$ as the composite $T1 \xrightarrow{!} 1 \xrightarrow{l} 1$.Then

- $e_X^\circ \boxplus e_Y^\circ \geq e_{X \times Y}^\circ$,
- $(r' \boxplus s') \circ (r \boxplus s) \leq (r' \circ r) \boxplus (s' \circ s)$.

Extending $\oplus$ to $\mathcal{T}$-Rel...

Let $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$ be $\mathcal{T}$-relations. We put $X \boxplus Y = X \times Y$ and define $r \boxplus s : X \times Y \nrightarrow X' \times Y'$ as

$$r \boxplus s = (r \oplus s) \cdot \tau_{X,Y}.$$

and $l_! : 1 \nrightarrow 1$ as the composite $T1 \xrightarrow{!} 1 \overset{l}{\nrightarrow} 1$.Then

- $e_X^\circ \boxplus e_Y^\circ \geq e_{X\times Y}^\circ$,
- $(r' \boxplus s') \circ (r \boxplus s) \leq (r' \circ r) \boxplus (s' \circ s)$.

For $(\_)_\# : \mathsf{V}\text{-Rel} \longrightarrow \mathcal{T}\text{-Rel}$ we have

- $(r \oplus r')_\# \leq r_\# \boxplus r'_\#$.
- $l_\# \leq l_!$.

... and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, I)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where*
$(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, I_!)$.

. . . and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, I)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where* $(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, I_!)$.

- For $(\_)_\# : \mathsf{V}\text{-Cat} \longrightarrow \mathcal{T}\text{-Cat}$ we have $\mathcal{T}$-functors

$$(X \oplus Y)_\# \longrightarrow X_\# \oplus Y_\# \qquad \text{and} \qquad E_\# \longrightarrow E.$$

. . . and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, I)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where* $(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, I_!)$.

- For $(\_)_\# : \mathsf{V}\text{-Cat} \longrightarrow \mathcal{T}\text{-Cat}$ we have $\mathcal{T}$-functors
$$(X \oplus Y)_\# \longrightarrow X_\# \oplus Y_\# \qquad \text{and} \qquad E_\# \longrightarrow E.$$
- For $\mathsf{S} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{V}\text{-Cat}$ we have $\mathcal{T}$-isomorphisms
$$\mathsf{S}(X \oplus Y) \longrightarrow \mathsf{S}(X) \oplus \mathsf{S}(Y) \qquad \text{and} \qquad \mathsf{S}(E) \longrightarrow E.$$

. . . and to $\mathcal{T}$-Cat

Theorem

Each monoidal structure $(\mathsf{V}, \oplus, I)$ *on* V *compatible with* $\mathcal{T}$ *defines a monoidal structure on* $\mathcal{T}$-Cat *where* $(X, a) \oplus (Y, b) = (X \times Y, a \boxplus b)$ *with neutral element* $E = (1, I_!)$.

- For $(\_)_\# : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ we have $\mathcal{T}$-functors

$$(X \oplus Y)_\# \longrightarrow X_\# \oplus Y_\# \qquad \text{and} \qquad E_\# \longrightarrow E.$$

- For $\mathsf{S} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ we have $\mathcal{T}$-isomorphisms

$$\mathsf{S}(X \oplus Y) \longrightarrow \mathsf{S}(X) \oplus \mathsf{S}(Y) \qquad \text{and} \qquad \mathsf{S}(E) \longrightarrow E.$$

- For $\mathsf{M} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ we have $\mathcal{T}$-functors

$$\tau_{X,Y} : \mathsf{M}(X \oplus Y) \longrightarrow \mathsf{M}(X) \oplus \mathsf{M}(Y) \quad \text{and} \quad ! : \mathsf{M}(E) \longrightarrow E.$$

Closedness of $\mathcal{T}$-Gph

Assume now that $u \oplus \_ : \mathsf{V} \to \mathsf{V}$ has right adjoint $u \multimap \_ : \mathsf{V} \to \mathsf{V}$.

Closedness of $\mathcal{T}$-Gph

Assume now that $u \oplus \_ : \mathsf{V} \to \mathsf{V}$ has right adjoint $u \multimap \_ : \mathsf{V} \to \mathsf{V}$.

Let $X = (X, a)$, $Y = (Y, b)$ be $\mathcal{T}$-graphs. Then

$$X \multimap Y = \{f : X \longrightarrow Y \mid f : X \oplus G \longrightarrow Y \text{ is a } \mathcal{T}\text{-functor}\}$$

(where $G = (1, e_X^\circ)$)

Closedness of $\mathcal{T}$-Gph

Assume now that $u \oplus \_ : \mathsf{V} \to \mathsf{V}$ has right adjoint $u \multimap \_ : \mathsf{V} \to \mathsf{V}$.

Let $X = (X, a)$, $Y = (Y, b)$ be $\mathcal{T}$-graphs. Then

$$X \multimap Y = \{f : X \longrightarrow Y \mid f : X \oplus G \longrightarrow Y \text{ is a } \mathcal{T}\text{-functor}\}$$

(where $G = (1, e_X^\circ)$) with structure

$$a \multimap b(\mathfrak{p}, h) = \bigwedge_{\substack{\mathfrak{q} \in T(X \times (X \multimap Y)), x \in X \\ \mathfrak{q} \longmapsto \mathfrak{p}}} (a(T\pi_X(\mathfrak{q}), x) \multimap b(T\mathrm{ev}(\mathfrak{q}), h(x))).$$

is a $\mathcal{T}$-graph as well. In fact, $X \oplus \_ \dashv X \multimap \_$.

Closed $\mathcal{T}$-categories

Lemma

$$
\begin{array}{ccc}
T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{\;T(\oplus)\;} & T\mathsf{V} \\
{\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\
\mathsf{V}\times\mathsf{V} & \xrightarrow[\;\oplus\;]{} & \mathsf{V}
\end{array}
\quad\Longrightarrow\quad
\begin{array}{ccc}
T(X\times Y) & \xrightarrow{\;\tau_{X,Y}\;} & TX\times TY \\
{\scriptstyle T_\xi(r\oplus s)}\big\downarrow\!\!\!\!\!\!- & & -\!\!\!\!\!\!\big\downarrow{\scriptstyle T_\xi r\oplus T_\xi s} \\
T(X'\times Y') & \xrightarrow[\;\tau_{X',Y'}\;]{} & TX'\times TY'.
\end{array}
$$

Closed $\mathcal{T}$-categories

Lemma

$$
\begin{array}{ccc}
T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\oplus)} & T\mathsf{V} \\
{\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\
\mathsf{V}\times\mathsf{V} & \xrightarrow[\oplus]{} & \mathsf{V}
\end{array}
\quad\Longrightarrow\quad
\begin{array}{ccc}
T(X\times Y) & \xrightarrow{\tau_{X,Y}} & TX\times TY \\
{\scriptstyle T_\xi(r\oplus s)}\downarrow & & \downarrow{\scriptstyle T_\xi r\oplus T_\xi s} \\
T(X'\times Y') & \xrightarrow[\tau_{X',Y'}]{} & TX'\times TY'.
\end{array}
$$

Theorem

$(\mathsf{V},\oplus,I)$ *closed, strictly compatible with* $\mathcal{T}$; $X=(X,a)\in\mathcal{T}$-Cat.

1. $a \multimap b$ *is transitive for each* $\mathcal{T}$*-category* $Y=(Y,b)$ *if*

$$(*)\quad \bigvee_{\mathfrak{x}\in TX} (T_\xi a(\mathfrak{X},\mathfrak{x})\oplus u)\otimes(a(\mathfrak{x},x_0)\oplus v)\geq a(m_X(\mathfrak{X}),x_0)\oplus(u\otimes v).$$

Closed $\mathcal{T}$-categories

Lemma

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\oplus)} & T\mathsf{V} \\ {\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\oplus]{} & \mathsf{V} \end{array} \quad\Longrightarrow\quad \begin{array}{ccc} T(X\times Y) & \xrightarrow{\tau_{X,Y}} & TX\times TY \\ {\scriptstyle T_\xi(r\oplus s)}\downarrow & & \downarrow{\scriptstyle T_\xi r\oplus T_\xi s} \\ T(X'\times Y') & \xrightarrow[\tau_{X',Y'}]{} & TX'\times TY'. \end{array}$$

Theorem

$(\mathsf{V},\oplus,I)$ *closed, strictly compatible with* $\mathcal{T}$; $X=(X,a)\in\mathcal{T}$-Cat.

1. $a\multimap b$ *is transitive for each* $\mathcal{T}$*-category* $Y=(Y,b)$ *if*

$$(*)\quad \bigvee_{\mathfrak{x}\in TX}(T_\xi a(\mathfrak{X},\mathfrak{x})\oplus u)\otimes(a(\mathfrak{x},x_0)\oplus v)\geq a(m_X(\mathfrak{X}),x_0)\oplus(u\otimes v).$$

2. *If* $a\multimap \hom_\xi$ *is transitive, then* $(*)$ *for all* $\mathfrak{X}\in T^2X$, $x_0\in X$ *and* $u,v\in\mathsf{V}$ *with* $\xi\cdot Tu=u\cdot !$ *and* $\xi\cdot Tv\leq v\cdot !$.

Closed $\mathcal{T}$-categories

Corollary

Consider $\oplus = \otimes$. *Let* $X = (X, a)$ *be a* $\mathcal{T}$*-category. Then*

1. *If* $a \cdot T_{\xi} a = a \cdot m_X$, *then* $\hom(a, b)$ *is transitive for each* $\mathcal{T}$*-category* $Y = (Y, b)$.
2. $a \cdot T_{\xi} a = a \cdot m_X$ *provided that* $\hom(a, \hom_{\xi})$ *is transitive.*

Closed $\mathcal{T}$-categories

Corollary

Consider $\oplus = \otimes$. *Let* $X = (X, a)$ *be a* $\mathcal{T}$*-category. Then*

1. *If* $a \cdot T_{\xi} a = a \cdot m_X$, *then* $\hom(a, b)$ *is transitive for each* $\mathcal{T}$*-category* $Y = (Y, b)$.
2. $a \cdot T_{\xi} a = a \cdot m_X$ *provided that* $\hom(a, \hom_{\xi})$ *is transitive.*
3. *Each Eilenberg-Moore algebra* (X, α) *is closed in* $\mathcal{T}$-Cat.

Closed $\mathcal{T}$-categories

Corollary

Consider $\oplus = \otimes$. Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

1. *If $a \cdot T_\xi a = a \cdot m_X$, then $\hom(a, b)$ is transitive for each $\mathcal{T}$-category $Y = (Y, b)$.*
2. *$a \cdot T_\xi a = a \cdot m_X$ provided that $\hom(a, \hom_\xi)$ is transitive.*
3. *Each Eilenberg-Moore algebra (X, α) is closed in $\mathcal{T}$-Cat.*
4. *If $Te_X \cdot e_X = m_X^\circ \cdot e_X$, then $X_\# = (X, r_\#)$ is closed for each V-category $X = (X, r)$.*

Compactness

- Degree of compactness: $\mathrm{comp}(X) = \bigwedge_{\mathfrak{x} \in TX} \bigvee_{x \in X} a(\mathfrak{x}, x)$.

Compactness

- Degree of compactness: $\mathrm{comp}(X) = \bigwedge_{\mathfrak{x}\in TX} \bigvee_{x\in X} a(\mathfrak{x}, x)$.
- X is $\oplus$-compact if $\mathrm{comp}(X) \geq I$

Compactness

- Degree of compactness: $\mathrm{comp}(X) = \bigwedge_{\mathfrak{x}\in TX} \bigvee_{x\in X} a(\mathfrak{x}, x)$.
- X is $\oplus$-compact if $\mathrm{comp}(X) \geq I$

Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. TFAE.

(i). *X is $\oplus$-compact.*

(ii). $\bigvee : (X \multimap \mathsf{V}) \longrightarrow \mathsf{V}$ *is a $\mathcal{T}$-functor (where $X \oplus \_ \dashv X \multimap \_$).*

(iii). $\gamma : |X|_I \longrightarrow \mathsf{V},\ \mathfrak{x} \longmapsto \bigvee_{x\in X} a(\mathfrak{x}, x)$ *is a $\mathcal{T}$-functor.*

Compactness

- Degree of compactness: $\mathrm{comp}(X) = \bigwedge_{\mathfrak{x}\in TX} \bigvee_{x\in X} a(\mathfrak{x}, x)$.
- X is $\oplus$-compact if $\mathrm{comp}(X) \geq I$

Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. TFAE.

(i). *X is $\oplus$-compact.*

(ii). *$\bigvee : (X \multimap \mathsf{V}) \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor (where $X \oplus \_ \dashv X \multimap \_$).*

(iii). *$\gamma : |X|_I \longrightarrow \mathsf{V},\ \mathfrak{x} \longmapsto \bigvee_{x\in X} a(\mathfrak{x}, x)$ is a $\mathcal{T}$-functor.*

Corollary

A $\mathcal{T}$-category $X = (X, a)$ is $\oplus$-compact iff $\pi_Y : Y \oplus X \longrightarrow Y$ is closed for each $\mathcal{T}$-category $Y = (Y, b)$.

$\mathcal{T}$-modules

A $\mathcal{T}$-module $\varphi : (X, a) \multimap (Y, b)$ is a $\mathcal{T}$-relation $\varphi : X \rightharpoondown\!\!\!\!\!\!+ \; Y$ such that

$$b \circ \varphi \leq \varphi \qquad \text{and} \qquad \varphi \circ a \leq \varphi.$$

$\mathcal{T}$-modules

A $\mathcal{T}$-module $\varphi : (X, a) \multimap (Y, b)$ is a $\mathcal{T}$-relation $\varphi : X \relbar\joinrel\mapsto Y$ such that

$$b \circ \varphi \leq \varphi \qquad \text{and} \qquad \varphi \circ a \leq \varphi.$$

Each $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ defines $\mathcal{T}$-modules $f_* \dashv f^*$:

$$f_* : (X, a) \multimap (Y, b);\ \ f_*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$$

$$f^* : (Y, b) \multimap (X, a);\ \ f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$$

$\mathcal{T}$-modules

A $\mathcal{T}$-module $\varphi : (X, a) \,-\!\!\circ\!\!\rightharpoonup (Y, b)$ is a $\mathcal{T}$-relation $\varphi : X \to\!\!\!\!\!\!\!\!\;\;| \;\; Y$ such that

$$b \circ \varphi \leq \varphi \qquad \text{and} \qquad \varphi \circ a \leq \varphi.$$

Each $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ defines $\mathcal{T}$-modules $f_* \dashv f^*$:

$$f_* : (X, a) \,-\!\!\circ\!\!\rightharpoonup (Y, b);\ \ f_*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$$

$$f^* : (Y, b) \,-\!\!\circ\!\!\rightharpoonup (X, a);\ \ f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$$

$f : (X, a) \longrightarrow (Y, b)$ is fully faithful iff $a = (\mathrm{id}_X)_* = f^* \circ f_*$.

Liftings and extensions

In V-Rel

For $\psi : X \relbar\joinrel\mapsto Z$, the composition maps

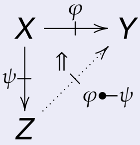
$$\_ \cdot \psi : \text{V-Rel}(Z, Y) \longrightarrow \text{V-Rel}(X, Y) \qquad \text{and}$$

$$\psi \cdot \_ : \text{V-Rel}(Y, X) \longrightarrow \text{V-Rel}(Y, Z)$$

have respective right adjoints

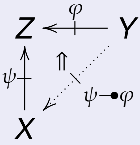
$$\_ \multimapdotinv \psi : \text{V-Rel}(X, Y) \longrightarrow \text{V-Rel}(Z, Y) \qquad \text{and}$$

$$\psi \multimapdot \_ : \text{V-Rel}(Y, Z) \longrightarrow \text{V-Rel}(Y, X).$$



(extension)

and



(lifting)

Liftings and extensions

In $\mathcal{T}$-Rel

For $\psi : X \relbar\joinrel\rightharpoonup Z$, the composition maps $\_ \circ \psi$ still has a right adjoint but $\psi \circ \_$ in general not.

Liftings and extensions

In $\mathcal{T}$-Rel

For $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Z$, the composition maps $\_ \circ \psi$ still has a right adjoint but $\psi \circ \_$ in general not. We pass from

$$
\begin{array}{ccc}
X & \overset{\varphi}{\relbar\joinrel\mapstochar\joinrel\rightharpoonup} & Y \\
{\scriptstyle \psi}\big\downarrow & & \\
Z & &
\end{array}
\qquad \text{to} \qquad
\begin{array}{ccc}
TX & \overset{\varphi}{\relbar\joinrel\mapstochar\joinrel\rightharpoonup} & Y \\
{\scriptstyle m_X^\circ}\big\downarrow & & \\
TTX & & \\
{\scriptstyle T_\xi \psi}\big\downarrow & & \\
TZ & &
\end{array}
$$

(in $\mathcal{T}$-Rel) (in V-Rel)

and define $\varphi \multimapinv \psi = \varphi \bullet\!\!- (T_\xi \psi \cdot m_X^\circ)$.

Modules as functors

The dual $\mathcal{T}$-category X^{op} of $X = (X, a)$ is defined as

$$X^{\mathrm{op}} = (\mathrm{M}(X)^{\mathrm{op}})_{\#}.$$

Modules as functors

The dual $\mathcal{T}$-category X^{op} of $X = (X, a)$ is defined as

$$X^{\mathrm{op}} = (\mathsf{M}(X)^{\mathrm{op}})_{\#}.$$

Theorem

For $\mathcal{T}$-categories (X, a) and (Y, b), and a $\mathcal{T}$-relation $\psi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$, the following assertions are equivalent.

i. $\psi : (X, a) \relbar\joinrel\circ\joinrel\rightharpoonup (Y, b)$ *is a $\mathcal{T}$-module.*

ii. *Both $\psi : |X| \otimes Y \longrightarrow \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ are $\mathcal{T}$-functors.*

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\begin{aligned} \alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) &\longrightarrow \mathcal{T}\text{-Map}(Y, X). \\ f &\longmapsto f_* \end{aligned}$$

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) \longrightarrow \mathcal{T}\text{-Map}(Y, X).$$
$$f \longmapsto f_*$$

We call a $\mathcal{T}$-category X

- L-separated if $\alpha_{Y,X}$ is injective, for all $\mathcal{T}$-categories Y.

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) \longrightarrow \mathcal{T}\text{-Map}(Y, X).$$
$$f \longmapsto f_*$$

We call a $\mathcal{T}$-category X

- L-separated if $\alpha_{Y,X}$ is injective, for all $\mathcal{T}$-categories Y.
- L-complete if $\alpha_{Y,X}$ is surjective, for all $\mathcal{T}$-categories Y.

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) \longrightarrow \mathcal{T}\text{-Map}(Y, X).$$
$$f \longmapsto f_*$$

We call a $\mathcal{T}$-category X

- L-separated if $\alpha_{Y,X}$ is injective, for all $\mathcal{T}$-categories Y.
- L-complete if $\alpha_{Y,X}$ is surjective, for all $\mathcal{T}$-categories Y.

Note: It is enough to consider $Y = G = (1, e_1^\circ)$.

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) \longrightarrow \mathcal{T}\text{-Map}(Y, X).$$
$$f \longmapsto f_*$$

We call a $\mathcal{T}$-category X

- L-separated if $\alpha_{Y,X}$ is injective, for all $\mathcal{T}$-categories Y.
- L-complete if $\alpha_{Y,X}$ is surjective, for all $\mathcal{T}$-categories Y.

Note: It is enough to consider $Y = G = (1, e_1^\circ)$.

Examples

- In Met: L-complete=Cauchy-complete.

L-separatedness/L-completeness

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories. We consider

$$\alpha_{Y,X} : \mathcal{T}\text{-Cat}(Y, X) \longrightarrow \mathcal{T}\text{-Map}(Y, X).$$
$$f \longmapsto f_*$$

We call a $\mathcal{T}$-category X

- L-separated if $\alpha_{Y,X}$ is injective, for all $\mathcal{T}$-categories Y.
- L-complete if $\alpha_{Y,X}$ is surjective, for all $\mathcal{T}$-categories Y.

Note: It is enough to consider $Y = G = (1, e_1^\circ)$.

Examples

- In Met: L-complete=Cauchy-complete.
- In Top: L-complete=weakly sober.

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
- $\varphi : 1 \multimap X$ is essentially a closed subset $A \subseteq X$.

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
- $\varphi : 1 \multimap X$ is essentially a closed subset $A \subseteq X$.
- $\psi : X \multimap 1$ is essentially a Zariski- and down-closed subset $\mathcal{A} \subseteq UX$.

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
- $\varphi : 1 \multimap X$ is essentially a closed subset $A \subseteq X$.
- $\psi : X \multimap 1$ is essentially a Zariski- and down-closed subset $\mathcal{A} \subseteq UX$.
- $\varphi \dashv \psi \iff \mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \,.\, \mathfrak{x} \to x\} \;\&\; \exists \mathfrak{x} \in \mathcal{A} \,.\, A \in \mathfrak{x}$

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
- $\varphi : 1 \multimapdotinv X$ is essentially a closed subset $A \subseteq X$.
- $\psi : X \multimapdotinv 1$ is essentially a Zariski- and down-closed subset $\mathcal{A} \subseteq UX$.
- $\varphi \dashv \psi \iff \mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \,.\, \mathfrak{x} \to x\} \;\&\; \exists \mathfrak{x} \in \mathcal{A} \,.\, A \in \mathfrak{x}$

Therefore

$$\begin{aligned} \varphi \text{ is left adjoint} &\iff \exists \mathfrak{x} \in UX \,.\, (A \in \mathfrak{x} \;\&\; \mathfrak{x} \to A) \\ &\iff A \text{ is irreducible.} \end{aligned}$$

Example: Top

Let X be a topological space. Then

- $M(X) = (UX, \leq)$ where $\mathfrak{x} \leq \mathfrak{y}$ if $\overline{\mathfrak{x}} \subseteq \mathfrak{y}$.
- $\varphi : 1 \multimapdotinv X$ is essentially a closed subset $A \subseteq X$.
- $\psi : X \multimapdotinv 1$ is essentially a Zariski- and down-closed subset $\mathcal{A} \subseteq UX$.
- $\varphi \dashv \psi \iff \mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \,.\, \mathfrak{x} \to x\} \;\&\; \exists \mathfrak{x} \in \mathcal{A} \,.\, A \in \mathfrak{x}$

Therefore

$$\begin{aligned} \varphi \text{ is left adjoint} &\iff \exists \mathfrak{x} \in UX \,.\, (A \in \mathfrak{x} \;\&\; \mathfrak{x} \to A) \\ &\iff A \text{ is irreducible.} \end{aligned}$$

and

$$\varphi \text{ is representable by } x \iff A = \overline{\{x\}}.$$

The Yoneda Lemma

For a $\mathcal{T}$-category $X = (X, a)$, both

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$$

are $\mathcal{T}$-functors.

The Yoneda Lemma

For a $\mathcal{T}$-category $X = (X, a)$, both

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$$

are $\mathcal{T}$-functors. Hence we have the Yoneda functor $y : X \longrightarrow \mathsf{V}^{|X|}$ (and – less important – also $y_w : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$).

The Yoneda Lemma

For a $\mathcal{T}$-category $X = (X, a)$, both

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$$

are $\mathcal{T}$-functors. Hence we have the Yoneda functor $y : X \longrightarrow \mathsf{V}^{|X|}$ (and – less important – also $y_w : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$).

Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

1. *For all $\mathfrak{x} \in TX$ and $\psi \in \mathsf{V}^{|X|}$, $[\![Ty(\mathfrak{x}), \psi]\!] \leq \psi(\mathfrak{x})$.*

The Yoneda Lemma

For a $\mathcal{T}$-category $X = (X, a)$, both

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$$

are $\mathcal{T}$-functors. Hence we have the Yoneda functor $y : X \longrightarrow \mathsf{V}^{|X|}$ (and – less important – also $y_w : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$).

Theorem

Let $X = (X, a)$ be a $\mathcal{T}$-category. Then

1. *For all $\mathfrak{x} \in TX$ and $\psi \in \mathsf{V}^{|X|}$, $[\![Ty(\mathfrak{x}), \psi]\!] \leq \psi(\mathfrak{x})$.*
2. *Let $\psi \in \mathsf{V}^{|X|}$. Then*

$$\forall \mathfrak{x} \in TX \,.\, \psi(\mathfrak{x}) \leq [\![Ty(\mathfrak{x}), \psi]\!] \iff \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor.}$$

The Yoneda embedding

We put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $\mathsf{V}^{|X|}$.

If $T1 = 1$, we have a fully faithful functor $y : X \longrightarrow \hat{X}$.

The Yoneda embedding

We put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $\mathsf{V}^{|X|}$.

If $T1 = 1$, we have a fully faithful functor $y : X \longrightarrow \hat{X}$.

Remarks

- In V-Cat we have $\hat{X} = \mathsf{V}^{X^{\mathrm{op}}}$.

The Yoneda embedding

We put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $\mathsf{V}^{|X|}$.

If $T1 = 1$, we have a fully faithful functor $y : X \longrightarrow \hat{X}$.

Remarks

- In V-Cat we have $\hat{X} = \mathsf{V}^{X^{\mathrm{op}}}$.
- However, $y_w : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$ is not fully faithful in Top.

The Yoneda embedding

We put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $\mathsf{V}^{|X|}$.

If $T1 = 1$, we have a fully faithful functor $y : X \longrightarrow \hat{X}$.

Remarks

- In V-Cat we have $\hat{X} = \mathsf{V}^{X^{\mathrm{op}}}$.
- However, $y_w : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}}$ is not fully faithful in Top.

From now on we assume $T1 = 1$.

L-closure

Definition

Let $X = (X, a)$ be a $\mathcal{T}$-category. For $M \subseteq X$ we define

$$\overline{M} = \{x \in X \mid i^* \circ x_* \dashv x^* \circ i_*\}.$$

and call $\overline{M}$ the L-closure of M.

L-closure

Definition

Let $X = (X, a)$ be a $\mathcal{T}$-category. For $M \subseteq X$ we define

$$\overline{M} = \{x \in X \mid i^* \circ x_* \dashv x^* \circ i_*\}.$$

and call $\overline{M}$ the L-closure of M.

Theorem

Then the following assertions are equivalent.

i. $x \in \overline{M}$.
ii. *For all $\mathcal{T}$-functors $\varphi, \psi : X \longrightarrow Y$ with L-separated codomain: if $\varphi|_M = \psi|_M$, then $\varphi(x) = \psi(x)$.*
iii. *For all $\mathcal{T}$-functors $\varphi, \psi : X \longrightarrow \mathsf{V}$: if $\varphi|_M = \psi|_M$, then $\varphi(x) = \psi(x)$.*

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.
- X L-complete, $M \subseteq X$ L-closed $\Rightarrow$ M is L-complete.

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.
- X L-complete, $M \subseteq X$ L-closed $\Rightarrow$ M is L-complete.
- X L-separated, $M \subseteq X$ L-complete $\Rightarrow$ M is L-closed.

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.
- X L-complete, $M \subseteq X$ L-closed $\Rightarrow$ M is L-complete.
- X L-separated, $M \subseteq X$ L-complete $\Rightarrow$ M is L-closed.
- $\hat{X}$ is closed in $\mathrm{V}^{|X|}$.

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.
- X L-complete, $M \subseteq X$ L-closed $\Rightarrow$ M is L-complete.
- X L-separated, $M \subseteq X$ L-complete $\Rightarrow$ M is L-closed.
- $\hat{X}$ is closed in $\mathsf{V}^{|X|}$.

Proposition

$\psi \in \hat{X}$ is a right adjoint $\mathcal{T}$-module if and only if $\psi \in \overline{y[X]}$.

L-closure

Further properties

- $f : X \longrightarrow Y$ is L-dense iff $f_* \circ f^* = (\mathrm{id}_Y)_* = b$.
- X L-complete, $M \subseteq X$ L-closed $\Rightarrow M$ is L-complete.
- X L-separated, $M \subseteq X$ L-complete $\Rightarrow M$ is L-closed.
- $\hat{X}$ is closed in $\mathsf{V}^{|X|}$.

Proposition

$\psi \in \hat{X}$ is a right adjoint $\mathcal{T}$-module if and only if $\psi \in \overline{y[X]}$.

Proof.

$\ldots \varphi = (\mathrm{id}_X)_* \circ\!\!- \psi$ and observe that $\varphi(x) = \hat{a}(e_{\hat{X}}(\psi)\, y(x))$ and

$$\xi \cdot T\varphi(\mathfrak{x}) = T_{\xi}\hat{a}(Te_{\hat{X}} \cdot e_{\hat{X}}(\psi), Ty(\mathfrak{x})) \ldots \quad \square$$

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

Theorem

The following assertions are equivalent.

i. *X is L-complete.*

ii. *X is injective with respect to fully faithful dense $\mathcal{T}$-functor.*

iii. *$y : X \longrightarrow \tilde{X}$ has a left inverse $\mathcal{T}$-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong \mathrm{id}_X$.*

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

Theorem

The following assertions are equivalent.

i. *X is L-complete.*
ii. *X is injective with respect to fully faithful dense $\mathcal{T}$-functor.*
iii. *$y : X \longrightarrow \tilde{X}$ has a left inverse $\mathcal{T}$-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong \mathrm{id}_X$.*

- V is injective w.r.t. fully faithful $\mathcal{T}$-functors.

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

Theorem

The following assertions are equivalent.

i. *X is L-complete.*

ii. *X is injective with respect to fully faithful dense $\mathcal{T}$-functor.*

iii. *$y : X \longrightarrow \tilde{X}$ has a left inverse $\mathcal{T}$-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong \mathrm{id}_X$.*

- V is injective w.r.t. fully faithful $\mathcal{T}$-functors.
- X with $a \cdot T_{\xi} a = a \cdot m_X$, Y L-complete $\Rightarrow$ Y^X L-complete.

L-completeness

We put $\tilde{X} = \overline{y[X]}$, then $y : X \longrightarrow \tilde{X}$ is fully faithful and dense.

Theorem

The following assertions are equivalent.

i. *X is L-complete.*
ii. *X is injective with respect to fully faithful dense $\mathcal{T}$-functor.*
iii. *$y : X \longrightarrow \tilde{X}$ has a left inverse $\mathcal{T}$-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong \mathrm{id}_X$.*

- V is injective w.r.t. fully faithful $\mathcal{T}$-functors.
- X with $a \cdot T_{\xi} a = a \cdot m_X$, Y L-complete $\Rightarrow$ Y^X L-complete.
- $\mathrm{V}^{|X|}$, $\hat{X}$, $\tilde{X}$ are L-complete.