FIBRATIONS IN LOGIC

Category Theory 2007

Portugal

June 2007

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HYPERDOCTRINES

Lawvere: Adjointness in Foundations (Dialectica 1969)

$$\mathbb{P} \downarrow \mathbb{T}$$

$\mathbb{P} \to \mathbb{T}$: cloven fibration

$\mathbb{T}$: category of types

fibre $\mathbb{P}(T)$: attributes of type T

- cartesian closed
- $\Sigma_f \dashv f^* \dashv \Pi_f$

+ good conditions

CATEGORICAL PROOF THEORY

Hyperdoctrines

(Lawvere)

Deduction Systems

↓

Free categories

(Lambek)

Type Theory

↕

LCC cat'ys

(Seely)

Indexed logic

Realizability toposes

Here a gap!

Structural proof theory

Linear logic

Differential λ-calculus

Knots

Cobordisms

CODOMAIN HYPERDOCTRINE

$\mathbb{T}$ is locally cartesian closed just when setting

$$\mathbb{P}(T) = \mathbb{T}/T$$

gives a hyperdoctrine.

This corresponds to (extensional) type theory:

$$\begin{array}{ccc} X & & Y \\ \downarrow & & \downarrow \\ S & \xrightarrow[f]{} & T \end{array}$$

$f^*\begin{pmatrix} Y \\ \downarrow \\ T \end{pmatrix}$ via pb $\qquad f^*\begin{pmatrix} Y \\ \downarrow \\ T \end{pmatrix}(s) = Y(f(s))$

$\Sigma_f\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix}$ by composition

$$\Sigma_f\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix}(t) = \Sigma s \in S(t).\ X(s)$$

$\Pi_f\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix}$ "sections"

$$\Pi_f\begin{pmatrix} X \\ \downarrow \\ S \end{pmatrix}(t) = \Pi s \in S(t).\ X(s)$$

AXIOM OF CHOICE

$$\begin{array}{lcl} C & = & \Sigma b \in B.\ C(b) \\ \downarrow & & \\ B & = & \Sigma a \in A.\ B(a) \\ \downarrow & & \\ A & & \\ \downarrow & & \\ 1 & & \end{array}$$

$$\Pi a \in A.\ \Sigma b \in B(a).\ C(b)$$

$$\longrightarrow \Sigma f : (\Pi a \in A.\ B(a)).\ \Pi a \in A.\ C(fa)$$

holds in that the canonical map

$$\Sigma f : (\Pi a \in A.\ Ba).\ \Pi a \in A\ C(f(a))$$

$$\longrightarrow$$

$$\Pi a \in A\ \Sigma b \in B(a).\ C(b)$$

is an isomorphism.

NON-STANDARD LOGICS

Fibration $\mathbb{P} \longrightarrow \mathbb{T}$ of propositions

- the fibres $\mathbb{P}(T)$ preordered

$$\phi \vdash \psi$$

and with logical structure

(connectives)

- the adjoints

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

are quantifiers

$$\exists_f \dashv f^* \dashv \forall_f$$

(preserved under substitution)

NON-STANDARD PROOFS

Fibration $\mathbb{P} \to \mathbb{T}$ of propositions and proofs

- a map $f : \phi \to \psi$ in $\mathbb{P}(T)$ is a proof of ψ from ϕ in context T
- the adjoints

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

act as quantifiers on both propositions and proofs

(preserved under substitution)

PROPOSITIONS FROM TYPES

Success of non-standard logic distracts from non-standard proof theory. But

Proof theory ⟶ Logic

Preordered reflection

[Already in Lawvere 1969]

Good methodology

"Curry – Howard Correspondence".

GÖDEL'S DIALECTICA INTERPRETATION

Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes
(Dialectica 1958)
(lectures in 1941 / 1942.)

To $\phi(\underline{a})$ formula of arithmetic associate

$$\phi^D(\underline{a}) = \exists u \, \forall x \, \phi_D(u, x; \underline{a})$$

in language with higher types, such that

if $HA \vdash \phi$ then for some terms $f(\underline{a})$ in Gödel's theory of primitive recursive functions

$$\vdash \forall x \, \phi_D(f(\underline{a}), x; \underline{a})$$

"Consistency Proof"

[See also Bishop 1970]

EXTRACTING INFORMATION FROM PROOFS

The main point of functional interpretations.

Recent explosion of work in logic

Kohlenbach.

Ferreira - Oliva

Avigad

But it looks a mess.

INTERPRETING IMPLICATION

Let $\phi^D = \exists u \, \forall x \; \phi_D(u,x)$

$\psi^D = \exists v \, \forall y \; \psi_D(v,y)$

Then

$(\phi \longrightarrow \psi)^D =$

$$\exists f \, \exists F \, \forall u \, \forall y . \; \phi_D(u, F(u,y)) \longrightarrow \psi_D(f(u),y)$$

with

$f : U \longrightarrow V$

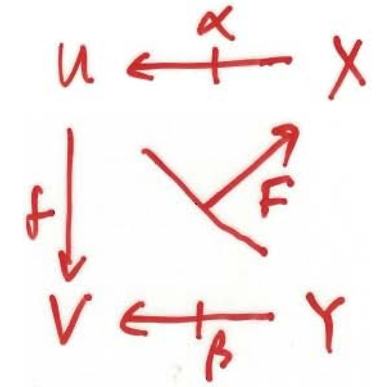
$F : U \times Y \longrightarrow X$

DIALECTICA CATEGORIES

Objects $U \overset{\alpha}{\longleftarrow\!\!\!+\!\!\!-} X$

(relations between U and X)

Maps



($f : U \longrightarrow V$

$F : U \times Y \longrightarrow X$ such that

$$\alpha(u, F(u,y)) \vdash_{u,y} \beta(f(u), y) \;)$$

WARM-UP : $\mathbb{C} \times \mathbb{C}^{op}$

Take $\mathbb{C}$ symmetric monoidal closed with finite products.
Then $\mathbb{C} \times \mathbb{C}^{op}$ is $*$-autonomous.

$$(U, X) \otimes (V, Y) = (U \otimes V,\ V \multimap X \times U \multimap Y)$$

$$(V, Y) \multimap (W, Z) = (V \multimap W \times Z \multimap Y,\ V \otimes Z)$$

If in addition $\mathbb{C}$ has sums (coproducts)
then $\mathbb{C} \times \mathbb{C}^{op}$ has products (and sums).

$$(U, X) \times (V, Y) = (U \times V,\ X + Y)$$

COMONADS ON $\mathbb{C} \times \mathbb{C}^{op}$

Simplification: $\mathbb{C}$ cartesian closed with +s.

Gödel's Dialectica Comonad

$$(U, X) \longmapsto (U, U \Rightarrow X)$$

Suppose $M(-)$ provides a notion of commutative monoid for $\mathbb{C}$, so

$$(U, X) \longmapsto (U, M(X)) \text{ comonad}$$

and suppose Gödel's distributes over it. Then we get

case $M(X)$ = finite multisets of $X = P_f(X)$

Diller–Nahm Comonad

$$(U, X) \longmapsto (U, U \Rightarrow P_f(X))$$

case $M(X) = 1$

Kreisel Comonad (Modified Realizability)

$$(U, X) \longmapsto (U, 1)$$

GIRARD TRANSLATION

Given $\mathbb{L}$ smc category with comonad ! take the Kleisli $\mathbb{K}$

$$\mathbb{K}(A,B) = \mathbb{L}(!A,B)$$

In good cases ~ $!A \otimes !B \cong !(A \times B)$

$\mathbb{K}$ is cartesian closed.

- Products as in $\mathbb{L}$

- $$\begin{aligned} \mathbb{K}(A \times B, C) &= \mathbb{L}(!(A \times B), C) \\ &\cong \mathbb{L}(!A \otimes !B, C) \\ &\cong \mathbb{L}(!A, !B \multimap C) \\ &= \mathbb{K}(A, !B \multimap C) \end{aligned}$$

So $B \Rightarrow C = !B \multimap C$ is the function space.

Works for Diller–Nahm

Modified Realizability

CHEAP RESULT

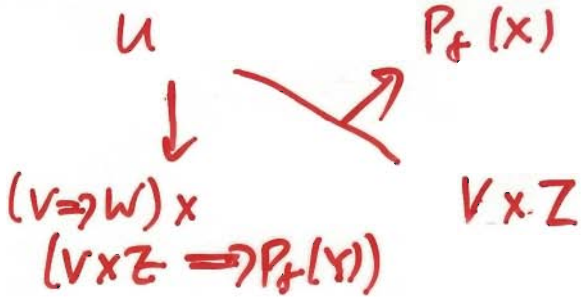
Every cartesian closed caty embeds fully + faithfully in the Kleisli category of a *-autonomous caty with comonad !

(Conservativity of the Girard translation of minimal intuitionistic logic into classical linear logic.)

DILLER – NAHM FUNCTION SPACE

$$(U, X) \longrightarrow ((V, Y) \Rightarrow (W, Z))$$

is

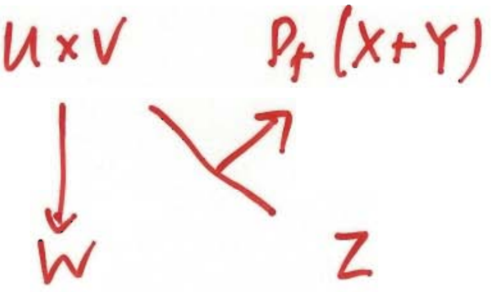


that is

$$U \times V \longrightarrow W$$
$$U \times V \times Z \longrightarrow P_f(Y)$$
$$U \times V \times Z \longrightarrow P_f(X)$$

$[P_f(Y) \times P_f(X) = P_f(Y + X)]$

that is



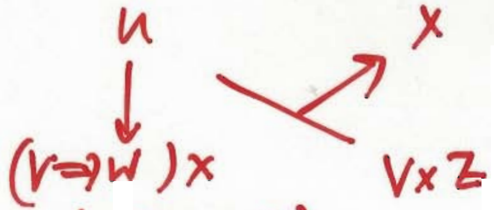
so it is

$$(U, X) \times (V, Y) \longrightarrow (W, Z).$$

GÖDEL FUNCTION SPACE

$$(U,X) \longrightarrow ((V,Y) \Rightarrow (W,Z))$$

is

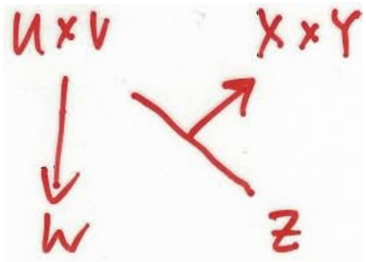


$$(V \times Z \Rightarrow Y)$$

that is

$$U \times V \longrightarrow W$$
$$U \times V \times Z \longrightarrow Y$$
$$U \times V \times Z \longrightarrow X$$

that is

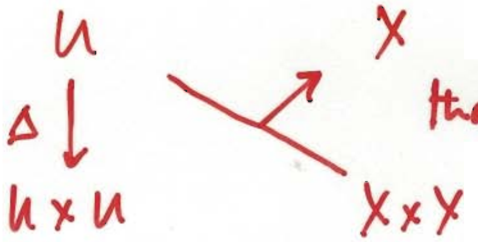


so it has has adjoint an 'unexpected' monoidal structure

$$(U,X) \bullet (V,Y) = (U \times V, X \times Y)$$

not a product.

WEAK DIAGONAL

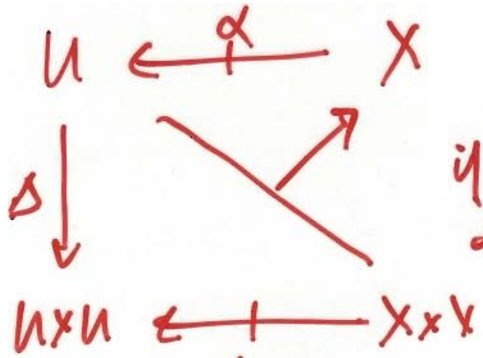


$thd : U \times X \times X \to X$

In the real interpretation we have propositions

$$U \xleftarrow{\ \alpha\ } X$$

and the weak diagonal uses definition by cases

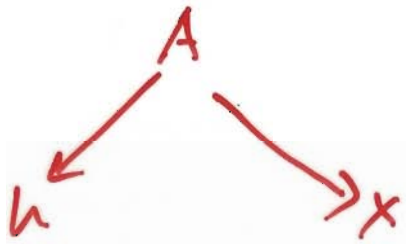


if $\alpha(u, x_1)$ then x_2 else x_1

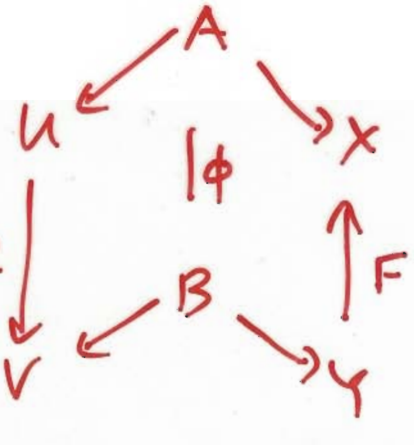
BASIC CONSTRUCTION

Take a locally cartesian closed category (with finite +s) get new category with

Objects



Maps



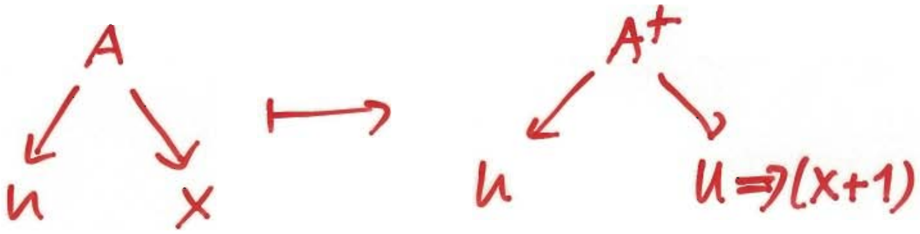
$f : U \to V$ $\quad$ $F : Y \to X$

$$\phi : \Pi u\, y.\ \big(A(u, F(y)) \Rightarrow B(fu, y)\big).$$

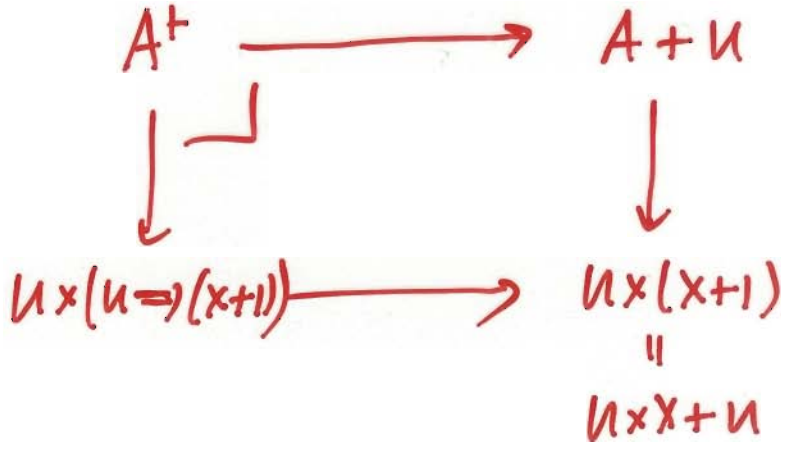
ERROR VARIANT

(of the Dialectica Interpretation)

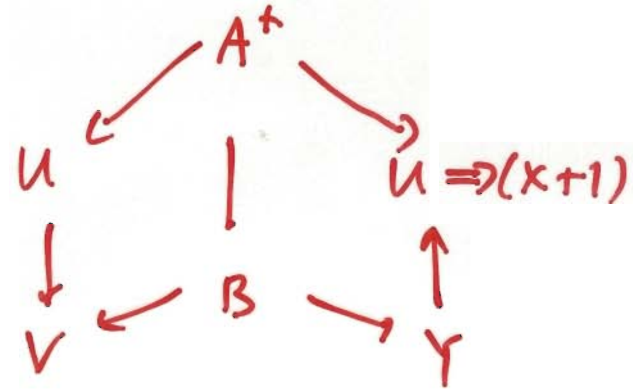
comes from



where



MAPS IN KLEISLI WITH ERRORS



consist of

$$f : U \longrightarrow V$$

$$F : U \times Y \longrightarrow X + 1$$

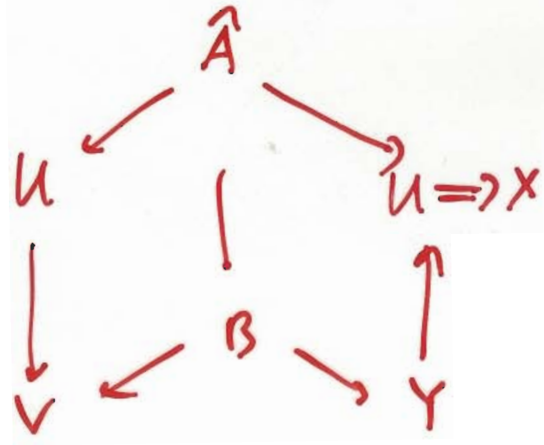
$$\phi : \prod_{uy} \begin{array}{c} |Fuy \in X| \Rightarrow A(u, Fuy) \Rightarrow B(fu, y) \\ \times \\ |Fuy \in 1| \Rightarrow B(fu, y) \end{array}$$

THEOREM

The Kleisli category $\mathbb{K}$ for the Error variant has products and a semi-closed structure for products.

So after splitting idempotents it gives a cartesian closed category.

CATEGORICAL FUNCTIONAL INTERPRETATION
(standard version)



consist of

$$f : U \longrightarrow V$$

$$F : U \times Y \longrightarrow X$$

$$\phi : \Pi uy \,.\, A(u, Fuy) \Rightarrow B(fu, y)$$

(Will not work as it stands but still worth understanding.)

GENERALIZED CATEGORICAL FUNCTIONAL INTERPRETATION

Idea: regard objects as

$$\begin{array}{ccc} A & & B \\ \downarrow & & \downarrow \\ U \times X & & V \times Y \\ \downarrow & & \downarrow \\ U & & V \end{array}$$

and replace them by

$$\begin{array}{ccc} X_2 & & Y_2 \\ \downarrow & & \downarrow \\ X_1 & & Y_1 \\ \downarrow & & \downarrow \\ X_0 & & Y_0 \end{array}$$

Now for maps

$$f_0 : X_0 \Longrightarrow Y_0$$

$$f_1 : \Pi x_0 \in X_0 .\ Y_1 f_0 x_0 \Longrightarrow X_1 x_0$$

$$f_2 : \Pi y_1 \in \left(\Sigma x_0 \in X_0 .\ Y_1 f_0 x_0\right)$$
$$X_2 f_1 y_1 \Longrightarrow Y_2(y_1)$$

(These specialize to previous defns.)

WHY STOP?

$$\begin{array}{ccc} \vdots & & \vdots \\ X_3 & & Y_3 \\ \downarrow & & \downarrow \\ X_2 & & Y_2 \\ \downarrow & & \downarrow \\ X_1 & & Y_1 \\ \downarrow & & \downarrow \\ X_0 & & Y_0 \end{array}$$

$$f_0 : X_0 \Rightarrow Y_0$$

$$f_1 : \Pi x_0 \in X_0 \,.\, Y_1\, f_0 x_0 \Rightarrow X_1\, x_0$$

$$f_2 : \Pi x_0 \in X_0 \,.\, y_1 \in Y_1 (f_0\, x_0) \,.\, X_2\, f_1 y_1 \Rightarrow Y_2\, y_1$$

$$f_3 : \Pi x_0 \,.\, y_1 \,.\, x_2 \in X_2 f_1 y_1 \,.\, Y_3\, f_2\, x_2 \Rightarrow X_3\, x_2$$

$$\vdots$$

What are these categories?

WHY STOP?

$$\begin{array}{ccc} \vdots & & \vdots \\ X_3 & & Y_3 \\ \downarrow & & \downarrow \\ X_2 & & Y_2 \\ \downarrow & & \downarrow \\ X_1 & & Y_1 \\ \downarrow & & \downarrow \\ X_0 & & Y_0 \end{array}$$

$$f_0 : X_0 \Rightarrow Y_0$$

$$f_1 : \Pi x_0 \in X_0 .\ Y_1\, f_0\, x_0 \Rightarrow X_1\, x_0$$

$$f_2 : \Pi x_0 \in X_0 . y_1 \in Y_1 (f_0\, x_0) .\ X_2\, f_1\, y_1 \Rightarrow Y_2\, y_1$$

$$f_3 : \Pi x_0 . y_1 . x_2 \in X_2\, f_1\, y_1 .\ Y_3\, f_2\, x_2 \Rightarrow X_3\, x_2$$

$$\vdots$$

What are these categories?

ENRICHMENT IN A LOCALLY CARTESIAN CLOSED

The formulae in terms of typing giving maps

$\Longrightarrow$

everything we do is enriched in our locally cartesian closed base category; and in an indexed fashion.

In that context we have categories with

sums Σ

products Π

and e.g. it makes sense freely to add Σs.

ABSTRACT CONSTRUCTION

$cod: \mathbb{T}^2 \longrightarrow \mathbb{T}$

fibre at 1 $X_0 \longrightarrow Y_0$

Now take the opposite (as fibred category) and freely add Σs

fibre at 1

$$X_1 \leftarrow\!\!-\!- Y_1$$

$$X_0 \longrightarrow Y_0$$

Now take the opposite of <u>this</u> (indexed enriched) category and freely add Σs

fibre at 1

$$X_2 \dashrightarrow Y_2$$

$$X_1 \leftarrow\!\!-\!- Y_1$$

$$X_0 \longrightarrow Y_0$$

ETC