Equivalence Relations and Subgroups

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Quantales

Recall that a (unital) quantale is a monoid object in the category of sup-lattices. More precisely, Q is a quantale if:

- For any two elements x and y, there is an element xy. This multiplication is associative, i.e. $(xy)z = x(yz)$ for all $x, y, z \in Q$ and has an identity, 1.
- Given any set of elements $\{x_i | i \in I\}$ in Q, there is a least upper bound $\bigvee_{i \in I} x_i$. (This implies that there is also a greatest lower bound for any set of elements.)
- Given any element y, and any set of elements $\{x_i | i \in I\}$, $y\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} yx_i$ and $\left(\bigvee_{i \in I} x_i\right) y = \bigvee_{i \in I} x_i y$.

Examples of Quantales

- Any **locale** is a quantale, with meet as multiplication.
- **The collection of relations on a set.** Multiplication is given by composition, i.e. $x \; RS \; y \Leftrightarrow (\exists z)(x \; S \; z \wedge z \; R \; y)$. Join is given by unions, where relations are viewed as subsets of $X \times X$.
- **The collection of subsets of a group.** Multiplication is pointwise – i.e. $AB = \{ab | a \in A, b \in B\}$. Join is union.
- **The collection of ideals of a C^*-algebra.**

Equivalence Relations

An equivalence relation E on X is a relation such that:

- E is reflexive, i.e. $1 \leqslant E$ in the quantale of relations on X.
- E is symmetric, i.e. if xEy then yEx.
- E is transitive, i.e. it is idempotent in the quantale of all relations.

Subgroups

A subset H of a group G is a subgroup if:

- H contains the identity, i.e. $1 \leqslant H$ in the quantale of all subsets of G.
- H is closed under taking inverses, i.e. if $x \in H$ then $x^{-1} \in H$.
- H is closed under multiplication, i.e. H is idempotent in the quantale of all subsets of G.

Embeddings

There are well-known embeddings between lattices of equivalence relations on a set and lattices of subgroups of a group.

- Given a group G, a subgroup induces an equivalence relation on the underlying set – relate two elements iff they are in the same left coset.
- Given an equivalence relation E on the set X, we form a subgroup of the group of permutations of X, namely the group of permutations that fix the equivalence classes.

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Do these embeddings come from some connection between the quantales of subsets of a group and relations on a set?

The Construction

Given a category $\mathcal{C}$, we can form a quantale $Q\mathcal{C}$ as follows:

- Elements are sets of morphisms in $\mathcal{C}$.
- Joins are unions.
- Multiplication is pointwise on elements that compose, i.e. $AB = \{fg | f \in A, g \in B, \text{dom } f = \text{cod } g\}$.

Examples of this Construction

$\mathcal{C}$	Q
Discrete category on X	Powerset of X
Group G	Quantale of subsets of G
Indiscrete category on X	Quantale of relations on X

Questions

- Given a quantale Q, under what circumstances can it be expressed as $Q\mathcal{C}$ for some category $\mathcal{C}$?
- When Q is $Q\mathcal{C}$ for some category $\mathcal{C}$, how can we reconstruct the category $\mathcal{C}$?

Finding the Category

- It is obvious that the morphisms of $\mathcal{C}$ will be exactly the indecomposable elements of $Q\mathcal{C}$. (i.e. elements that cannot be expressed as a join of strictly smaller elements.)
- We can obtain the objects of $\mathcal{C}$ as the identity morphisms, which are just the indecomposable elements that are $\leqslant 1$.

Ordered Categories

In fact it makes sense to generalise this construction to downsets of morphisms on ordered categories for the following reasons:

- When we construct the quantale from an unordered category $\mathcal{C}$, the indecomposable elements are all incomparable. This is an unnecessary extra condition on the quantale.
- There is an obvious embedding of the category of quantales into the category of ordered categories. This embedding is right adjoint to our downsets of morphisms construction.

Identities

When dealing with ordered categories, we need to be more careful in identifying which morphisms are identities. Downsets I generated by identity morphisms satisfy the following two equivalent conditions:

- $(\forall x \in Q\mathcal{C})(Ix = I\top \wedge x)$ and $(\forall x \in Q\mathcal{C})(xI = \top I \wedge x)$.
- $(\forall x, y \in Q\mathcal{C})(I(x \wedge y) = Ix \wedge y)$ and $(\forall x, y \in Q\mathcal{C})((x \wedge y)I = xI \wedge y)$.

We will call an element of an arbitrary quantale Q *objective* if it satisfies these properties. We will denote the collection of objective elements in Q by $\mathcal{I}\mathrm{d}_Q$. Where Q is obvious, we will omit the subscript.

Theorem

A quantale Q is the quantale of downsets of morphisms of a partially ordered category, if and only if the following conditions and their reverses (i.e. the conditions obtained by changing the order of all multiplications) hold:

1. *Q is a frame as a lattice. (Condition 2 then forces Q to be CCD.)*
2. *Q is generated by indecomposables as a* $\vee$*-semilattice.*
3. *All indecomposable objects* $x \in Q$ *have the property that the right adjoint* $x \to \_$ *to* $x.\_$ *preserves all inhabited joins.*

$\ldots$

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4. *The functions* $\top\_ : \mathcal{I}\mathrm{d} \longrightarrow Q$ *and* $\_\top : \mathcal{I}\mathrm{d} \longrightarrow Q$ *have left adjoints* dom *and* cod *respectively.*
5. dom *and* cod *satisfy the equations* $cod\,(fg) = cod\,(f\,cod\,(g))$ *and* $dom\,(fg) = dom\,(dom\,(f)g)$.

5'. *Equivalently, if* $g \leqslant i\top$ *and* $fg \leqslant j\top$*, for identities,* i *and* j*, then* $fi \leqslant j\top$.

6. *Every identity is a join of indecomposable identities.*

Functors

Given a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, what does this give between $Q\mathcal{C}$ and $Q\mathcal{D}$?

- It gives a sup-homomorphism $Q\mathcal{C} \xrightarrow{F_*} Q\mathcal{D}$, given by $F_*(A) = \{F(f) | f \in A\}$. This is a lax quantale homomorphism (i.e. $F_*(A)F_*(B) \leqslant F_*(AB)$ and $F_*(1) \leqslant 1$).

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- It also gives a lattice homomorphism $Q\mathcal{D} \xrightarrow{F^*} Q\mathcal{C}$, given by $F^*(A) = \{f \in \text{mor}\, \mathcal{C} | F(f) \in A\}$. This is adjoint to F_* as morphisms of ordered sets. It is therefore a colax quantale homomorphism.

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- Finally, there is a meet homomorphism $Q\mathcal{C} \xrightarrow{F^!} Q\mathcal{D}$, which is adjoint to F^*. It is given by $F^!(A) = \{f \in \text{mor } \mathcal{D} | (\forall g \in \text{mor } \mathcal{C})(F(g) = f \Rightarrow g \in A)\}$.

Embedding of Subgroups into Equivalence Relations

Given a group G, we have seen that:

- The quantale of subsets of G is the quantale of sets of morphisms of G as a 1-object category.
- The quantale of relations on the underlying set of G is the quantale of sets of morphisms in the indiscrete category $*\backslash G$.

There is a forgetful functor $*\backslash G \xrightarrow{F} G$. The embedding of lattices we saw earlier is just F^* for this functor, restricted to subgroups of G.

Quantale Homomorphisms

Given an order-preserving functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, when do F_* and F^* actually preserve the multiplication in their quantales?

- F_* preserves multiplication iff F has the property that given any composable morphisms $f, g \in \text{mor}\,\mathcal{C}$, and any $h \leqslant F(f)F(g)$ in mor $\mathcal{D}$, we can find $f' \leqslant f$ and $g' \leqslant g$ composable in mor $\mathcal{C}$, such that $h \leqslant F(f'g')$.
- F^* preserves multiplication iff F has the property that given a morphism $h \in \text{mor}\,\mathcal{C}$, and a composable pair of morphisms $f, g \in \text{mor}\,\mathcal{D}$, such that $F(h) \leqslant fg$, then we can find composable morphisms $f', g' \in \text{mor}\,\mathcal{C}$, such that $h \leqslant f'g'$, and $F(f') \leqslant f$, and $F(g') \leqslant g$.

These conditions are related to the ordered Conduché conditions.

Factorisation

- We can factor any ordered functor F into an ordered functor F_1 such that ${F_1}^*$ preserves multiplication, followed by an ordered functor F_2 such that ${F_2}_*$ preserves multiplication.
- This is related to the factorisation of an adjoint pair of a lax functor and a colax functor into an adjunction where the left adjoint is a pseudofunctor, followed by and adjunction where the right adjoint is a pseudofunctor.