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R = number line in suitable $\mathcal{E}$

$D = \{x \in R \mid x^2 = 0\}$

Neighbour relation $\sim$ on R:

$$x \sim y \quad \Leftrightarrow \quad (x-y)^2 = 0$$

On R^n:

$$\underline{x} \sim \underline{y} \quad \Leftrightarrow \quad \varphi(\underline{x}) \sim \varphi(\underline{y}) \text{ for all } \varphi : R^n \to R.$$

Reflexive, symmetric (not transitive)

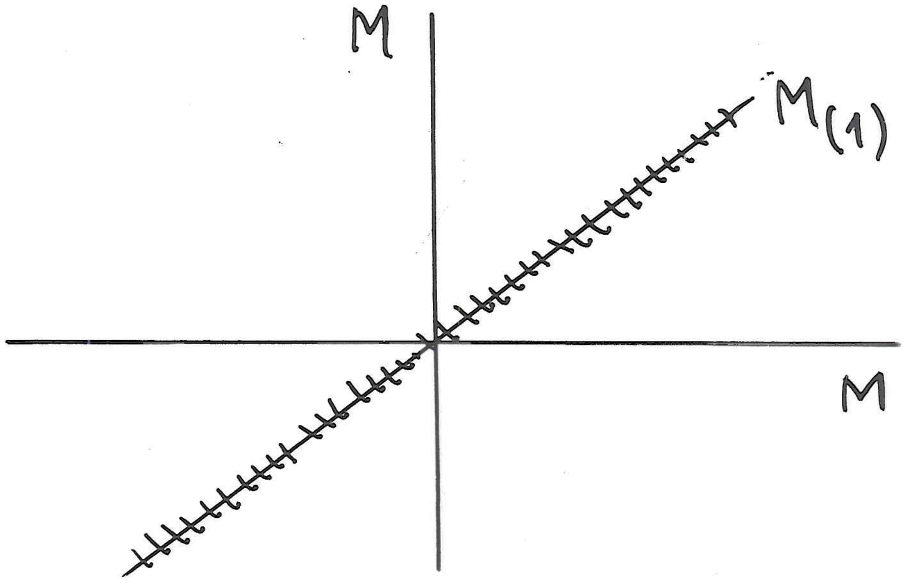
On any manifold M, have $\sim$.

$$M_{(1)} \subseteq M \times M$$
$$= \{(x,y) \in M \times M \mid x \sim y\}$$

"first neighbourhood of the diagonal"

In algebraic geometry: $(A \otimes A)/I^2$,

A = ring of functions on M (affine scheme)

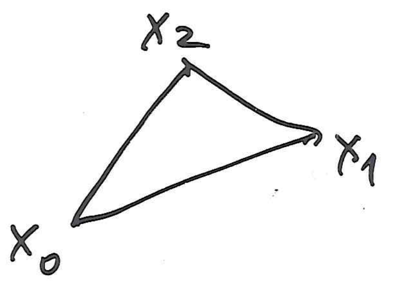


M manifold, $\quad M_{(1)} \subseteq M \times M$

Simplicial set $M_{(\bullet)}$:

$M_{(n)}$ = set of infinitesimal n-simplices

$$= \{(x_0, \dots, x_n) \mid x_i \sim x_j \quad \forall i,j\}$$



$\Rightarrow$ Theory of combinatorial differential forms

(K 1981, B&M 2001)

Affine combinations

$(x_0, \dots, x_n)$ infinitesimal n-simplex in M

Thm • Affine combinations of $x_0, \dots, x_n$ make sense

• All such combinations are mutual neighbours

• Any map preserves such affine combinations

Coroll Get map $R^n \xrightarrow{[\![x_0, \dots, x_n]\!]} M$

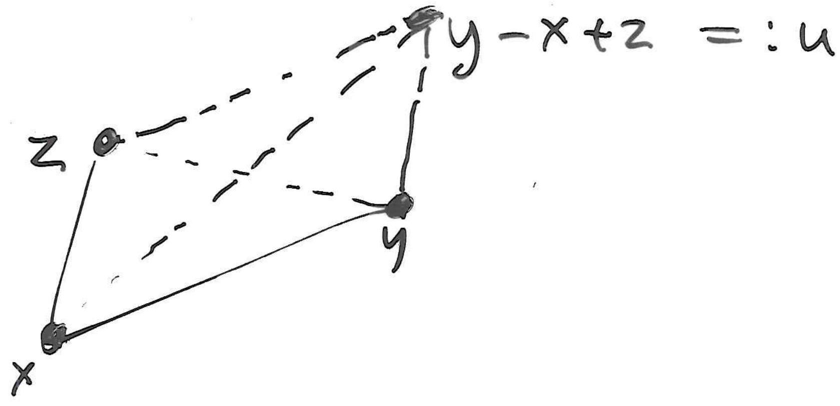
$$(t_1, \dots, t_n) \longmapsto (1 - \sum t_i) x_0 + t_1 x_1 + \dots + t_n x_n$$

Get also a 2^n-tuple of points in M: the values of $[\![x_0, \dots, x_n]\!]$ on the corners of $I^n \subset R^n$.

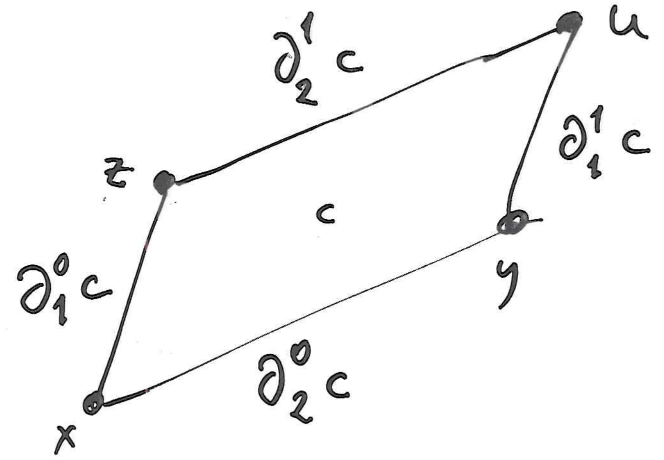
"Infinitesimal parallelepipedum"

Infinitesimal 2-simplex (x, y, z)

gives rise to inf. parallelogram



<u>Note:</u> if we do not assume $z \sim y$, this construction is a <u>structure</u> on M: "affine connection".



Get <u>cubical</u> <u>complex</u> of infinitesimal parallelepipeda in M.

$$M_{[\bullet]}$$

Has also <u>reversion</u> structure:

$$\rho_1[xyzu] = [y\,x\,u\,z]$$

$$\rho_2[xyzu] = [z\,u\,x\,y]$$

In dim. 1 $\quad \rho[xy] = [yx]$,

"$\sim$ is a <u>symmetric</u> relation".

Cubical complex <u>with</u> <u>reversions</u>

Cub_ω = cat. of <u>cubical sets</u> with reversion

$\{C_n\}$

Grd_ω = cat. of <u>cubical groupoids</u> :

each C_n is a gpd' in n ways :

$$C_n \underset{\partial_i^1}{\overset{\partial_i^0}{\rightrightarrows}} C_{n-1}$$

compatibly

Cubical gpd's have canonical reversions ρ_i

<u>Truncated versions</u>

Cub_n = cat. of n-cubical sets with reversions

Grd_n = cat of n-cubical groupoids

$$\mathrm{tr} : \mathrm{Cub}_{n+1} \longrightarrow \mathrm{Cub}_n$$

Has adjoints on both sides.

Example

Cub_1 = cat. of reflexive symmetric graphs

$$\rho \circlearrowright C_1 \underset{\partial^1}{\overset{\partial^0}{\underset{\xleftarrow{\ \varepsilon\ }}{\rightrightarrows}}} C_0$$

ε = reflexive, ρ = symmetric

Grd_1 = cat. of ordinary groupoids.

Given a manifold M,

and an n-cubical gpd. G with $G_0 = M$

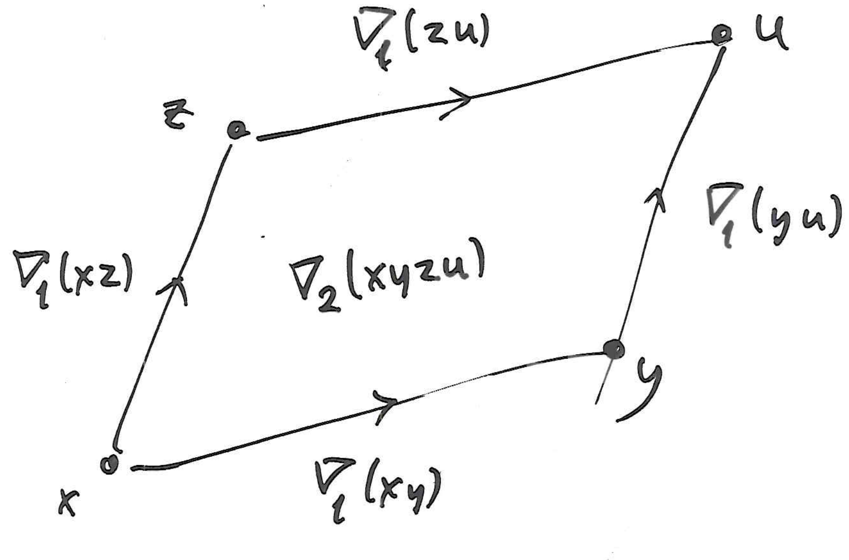
<u>Def.</u> An n-<u>connection</u> in G

is a morphism over M

$$M_{[\![n]\!]} \xrightarrow{\nabla} G$$

of n-cubical sets with reversion.

$n = 2$

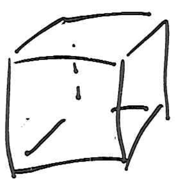


$$\begin{tikzcd} \mathrm{Cub}_{n+1} \arrow[r, shift left, "tr"] & \mathrm{Cub}_n \arrow[l, shift left, "(\ )'"] \end{tikzcd}$$

C'_{n+1} = set of $(n+1)$-shells in C

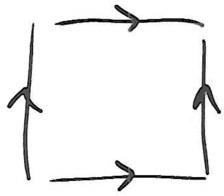
e.g. $n=2$:

C'_3 = set of 6-tuples of 2-cells which match



e.g. $n=1$:

C'_2 = set of 4-tuples of 1-cells which match



$(\ )'$ takes n-cubical <u>groupoids</u> to $(n+1)$-cubical <u>groupoids</u>

$$\mathrm{tr}\left(M_{[\![n+1]\!]}\right) = M_{[\![n]\!]}$$

Hence by adjointness $\mathrm{tr} \dashv (\ )'$

$$\frac{M_{[\![n]\!]} \xrightarrow{\nabla} G}{M_{[\![n+1]\!]} \xrightarrow{\hat{\nabla}} G'}$$

$(n+1)$ - connection $\hat{\nabla}$

$$M_{[n+1]} \xrightarrow{\hat{\hat{\nabla}}_{n+1}} (n+1)\text{-shells in } G$$

"formal curvature of ∇".

e.g. for $n=1$

$$\frac{M_{[\![1]\!]} \xrightarrow{\nabla} G}{M_{[\![2]\!]} \xrightarrow{\hat{\nabla}} G'}$$

$\hat{\nabla}[xyzu]$ = the 2-shell in G

$$\begin{tikzcd}
z \arrow[r, "\nabla(zu)"] & u \\
x \arrow[u, "\nabla(xz)"] \arrow[r, "\nabla(xy)"'] & y \arrow[u, "\nabla(yu)"']
\end{tikzcd}$$

<u>Curvature</u> = lack of commutativity of such shells

∇ <u>flat</u> = such squares commute

What does it mean for an $(n+1)$-shell to commute

Brown - Spencer - Higgins "connections" Γ on G

Converts n-cubical groupoids into crossed complexes ($\sim$ bundle over M of chain complexes, with an action)

$$G \xrightarrow{\Phi} Cr(G)$$

Flat ∇ : $\quad \Phi \circ \hat{\nabla} = 0$

Thm (Formal Bianchi identity)
If $\nabla : M_{[\![n]\!]} \to G$ is an n-connection, then $\hat{\nabla} : M_{[\![n+1]\!]} \to G'$ is a flat $n+1$-connection

Holonomy

$S_{[\cdot]}(M)$ = cubical complex with reversions

of singular cubes in M

$I^n \to M$ $\quad$ (or $R^n \to M$)

$$\begin{tikzcd} M_{[\![n]\!]} \arrow[r, "\nabla"] \arrow[d, hook, "i"'] & G \\ S_{[\![n]\!]}(M) \arrow[ur, dotted, "\int\nabla"'] & \end{tikzcd}$$

$i(x_0,\dots,x_n) = [\![x_0,\dots,x_n]\!] : R^n \to M$

Subdivision property / alternation property

to guarantee uniqueness of $\int\nabla$

$$\frac{S_{[\![n]\!]}(M) \xrightarrow{h} G}{S_{[\![n+1]\!]}(M) \xrightarrow{\hat{h}} G'}$$

Theorem (Formal Stokes')

$$\left(\int\nabla\right)^{\wedge} = \int\hat{\nabla}$$

Differential n-forms

Def'n A cubical-combinatorial n-form on M is a map

$$M_{[n]} \xrightarrow{\omega} R$$

taking value 0 on degenerate cubes.

Bijection between such, and differential n-forms

Cubical coboundary $\Longleftrightarrow$ exterior derivative

Can view such n-form as an n-connection in a suitable n-cubical gp'd $M_n(R)$ (with BSH-structure)

folded curvature $\Longleftrightarrow$ coboundary

$$(M_n(R))_k = M^{2^k} \qquad k < n$$

$$(M_n(R))_n = M^{2^n} \times R$$