Lawvere 2-theories

joint work with John Power

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Outline

Ordinary Lawvere theories

- Write $\mathcal{S}$ for the skeletal category of finite sete, and $J : \mathcal{S} \to \mathbf{Set}$ for the inclusion. $\mathcal{S}$ is the free category with finite coproducts/colimits on 1
- A (classical) *Lawvere theory* is an identity-on-objects functor $E : \mathcal{S}^{\mathrm{op}} \to \mathcal{L}$ which preserves finite products/limits. $\mathcal{L}$ will have all finite products but not necessarily all finite limits.
- A *model* of $\mathcal{L}$ is a functor $X : \mathcal{L} \to \mathbf{Set}$ which preserves finite products
- Equivalently for which $XE : \mathcal{S}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite products/limits . . . or equivalently for which $XE = \mathbf{Set}(J-, A)$ for some $A \in \mathbf{Set}$ (in fact $A = X1$)
- $\mathcal{L}(m, n) = \mathcal{L}(m, 1)^n$, where $\mathcal{L}(m, 1)$ is the set of m-ary operations

The category of models

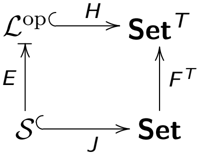
- Write $\mathrm{Mod}(\mathcal{L})$ for the category of models; the morphisms are natural transformations
- Pullback diagram

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{L}) & \hookrightarrow & [\mathcal{L}, \mathbf{Set}] \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle [E,\mathbf{Set}]} \\ \mathbf{Set} & \underset{\mathbf{Set}(J,1)}{\hookrightarrow} & [\mathcal{S}^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

  where $\mathbf{Set}(J,1)$ sends a set X to corresponding finite-product-preserving functor $\mathbf{Set}(J-,X) : \mathcal{S}^{\mathrm{op}} \to \mathbf{Set}$
- Forgetful functor U is monadic; thus every Lawvere theory determines a monad on $\mathbf{Set}$.

Finitary monads on **Set**

- A functor is *finitary* if it preserves filtered colimits. A monad is finitary if its underlying endofunctor is so. T : **Set** $\to$ **Set** is finitary iff it is the left Kan extension of $TJ : \mathcal{S} \to$ **Set**. Monads arising from Lawvere theories are finitary.
- Given a finitary monad T can form



  and now $E : \mathcal{S}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere theory, and $\mathbf{Set}^T$ is its category of models
- This gives an equivalence between Lawvere theories and finitary monads on **Set** [Linton].

The enriched version (Power)

- Version involving symmetric monoidal closed $\mathcal{V}$ in place of **Set**
- $\mathcal{V}$ should be *locally finitely presentable as a closed category* (Kelly) in order to have good notion of finite object of $\mathcal{V}$ (i.e. arity). Then use finite *cotensors* in place of finite products
- For $\mathcal{V}$-category $\mathcal{K}$, object $A \in \mathcal{K}$ and $X \in \mathcal{V}$, the cotensor A^X (sometimes called $X \pitchfork A$) defined by

$$\mathcal{K}(B, A^X) \cong \mathcal{V}(X, \mathcal{K}(B, A))$$

  Say that $\mathcal{K}$ has finite cotensors if A^X exists for all $A \in \mathcal{K}$ and all finitely presentable $X \in \mathcal{V}$
- e.g. if $\mathcal{V} = \textbf{Cat}$, can have operations with arity given by any finitely presentable category not just the discrete ones

General notion of theory

- Consider a symmetric monoidal closed LFP $\mathcal{V}$ as above and an LFP $\mathcal{V}$-category $\mathcal{K}$ i.e. $\mathcal{K} \simeq \mathrm{Lex}(\mathcal{K}_f^{\mathrm{op}}, \mathcal{V})$ for $J : \mathcal{K}_f \to \mathcal{K}$ the full subcategory of finitely presentable objects. Want notion of theory equivalent to finitary $\mathcal{V}$-monads on $\mathcal{K}$
- Given finitary $\mathcal{V}$-monad T, follow previous construction

$$\begin{array}{ccc} \mathcal{L}^{\mathrm{op}} & \xhookrightarrow{H} & \mathcal{K}^T \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle F^T} \\ \mathcal{K}_f & \xhookrightarrow[J]{} & \mathcal{K} \end{array}$$

- J preserves finite colimits, F^T preserves colimits, and H reflects colimits, so E preserves finite colimits

Definition (Nishizawa-Power)

A *Lawvere $\mathcal{K}$-theory* is an identity-on-objects, finite-limit-preserving $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$

General notion of model

- Monad T and induced theory $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$ as above
- Pullback diagram

$$\begin{array}{ccc} \mathcal{K}^T & \xrightarrow{\mathcal{K}^T(H,1)} & [\mathcal{L},\mathcal{V}] \\ {\scriptstyle U^T}\downarrow & & \downarrow{\scriptstyle [E,\mathcal{V}]} \\ \mathcal{K} & \xrightarrow{\mathcal{K}(J,1)} & [\mathcal{K}_f^{\mathrm{op}},\mathcal{V}] \end{array}$$

Definition (Nishizawa-Power)

The category of models of a theory $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$ is given by the pullback

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{L}) & \hookrightarrow & [\mathcal{L},\mathcal{V}] \\ \downarrow & & \downarrow{\scriptstyle [E,\mathcal{V}]} \\ \mathcal{K} & \xrightarrow{\mathcal{K}(J,1)} & [\mathcal{K}_f^{\mathrm{op}},\mathcal{V}] \end{array}$$

The equivalence between monads and theories

Theorem (Nishizawa-Power)

The category $\mathbf{Law}(\mathcal{K})$ *of Lawvere theories on* $\mathcal{K}$ *is equivalent to the category* $\mathbf{Mnd}_{\mathsf{f}}(\mathcal{K})$ *of finitary monads on* $\mathcal{K}$.

- finitary monad T gives theory $\Phi(T)$ given by $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$
- theory $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$ gives finitarily monadic $\mathrm{Mod}(\mathcal{L}) \to \mathcal{K}$ and so finitary monad $\Psi(\mathcal{L})$

$\Psi(\Phi(T)) \cong T$ follows from pullback

$$\begin{array}{ccc} \mathcal{K}^T & \hookrightarrow & [\mathcal{L},\mathcal{V}] \\ \downarrow & & \downarrow {\scriptstyle [E,\mathcal{V}]} \\ \mathcal{K} & \hookrightarrow & [\mathcal{K}_f^{\mathrm{op}},\mathcal{V}] \end{array}$$

$\Phi(\Psi(\mathcal{L})) \cong \mathcal{L}$ because Lan_E gives free models

$$\begin{array}{ccccc} \mathcal{L}^{\mathrm{op}} & \hookrightarrow & \mathrm{Mod}(\mathcal{L}) & \hookrightarrow & [\mathcal{L},\mathcal{V}] \\ {\scriptstyle E}\uparrow & & {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle \mathrm{Lan}_E} \\ \mathcal{K}_f & \hookrightarrow & \mathcal{K} & \hookrightarrow & [\mathcal{K}_f^{\mathrm{op}},\mathcal{V}] \end{array}$$

with $Y : \mathcal{L}^{\mathrm{op}} \hookrightarrow [\mathcal{L},\mathcal{V}]$ the composite along the top.

Models in other categories

- Since $\mathcal{K} \simeq \mathrm{Lex}(\mathcal{K}_f^{\mathrm{op}}, \mathcal{V})$ we can equivalently define models via the pullback

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{L}) & \hookrightarrow & [\mathcal{L}, \mathcal{V}] \\ \downarrow & & \downarrow {\scriptstyle [E,\mathcal{V}]} \\ \mathrm{Lex}(\mathcal{K}_f^{\mathrm{op}}, \mathcal{V}) & \hookrightarrow & [\mathcal{K}_f^{\mathrm{op}}, \mathcal{V}] \end{array}$$

- If $\mathcal{A}$ has finite limits, then define category of models in $\mathcal{A}$ by the pullback

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{L}, \mathcal{A}) & \hookrightarrow & [\mathcal{L}, \mathcal{A}] \\ \downarrow & & \downarrow {\scriptstyle [E,\mathcal{A}]} \\ \mathrm{Lex}(\mathcal{K}_f^{\mathrm{op}}, \mathcal{A}) & \hookrightarrow & [\mathcal{K}_f^{\mathrm{op}}, \mathcal{A}] \end{array}$$

- Thus a model is a functor $X : \mathcal{L} \to \mathcal{A}$ whose restriction $XE : \mathcal{K}_f^{\mathrm{op}} \to \mathcal{A}$ along E preserves finite limits

Models as right adjoint functors

- $E : \mathcal{K}_f^{\mathrm{op}} \longrightarrow \mathcal{L}$ theory
- $\mathcal{A}$ category with limits
- We'll consider models of $\mathcal{L}$ in $\mathcal{A}$

$$\frac{\mathcal{L} \xrightarrow{X} \mathcal{A}}{\dfrac{\mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{op}}}{\dfrac{[\mathcal{L},\mathcal{V}] \underset{\text{left adj.}}{\longrightarrow} \mathcal{A}^{\mathrm{op}}}{\mathcal{A}^{\mathrm{op}} \underset{\text{right adj.}}{\xrightarrow{\mathcal{A}(1,X)}} [\mathcal{L},\mathcal{V}]}}}$$

X is a model
iff
$\mathcal{A}(1,X) : \mathcal{A}^{\mathrm{op}} \to [\mathcal{L},\mathcal{V}]$
lands in $\mathsf{Mod}(\mathcal{L})$.

$$\frac{\mathcal{L} \underset{\text{model}}{\longrightarrow} \mathcal{A}}{\dfrac{\mathcal{A}^{\mathrm{op}} \underset{\text{right adj.}}{\longrightarrow} \mathsf{Mod}(\mathcal{L})}{\dfrac{\mathsf{Mod}(\mathcal{L}) \underset{\text{left adj.}}{\longrightarrow} \mathcal{A}^{\mathrm{op}}}{\mathsf{Mod}(\mathcal{L})^{\mathrm{op}} \underset{\text{right adj.}}{\longrightarrow} \mathcal{A}}}}$$

- and so $\mathsf{Mod}(\mathcal{L},\mathcal{A}) \simeq \mathsf{Radj}(\mathsf{Mod}(\mathcal{L})^{\mathrm{op}},\mathcal{A})$
  (cf Kelly notion of comodel wrt dense functor.)

The theory of theories

- $\mathcal{K}$ is LFP $\mathcal{V}$-category
- $|\mathcal{K}_f|$ set of objects of (a skeleton of) $\mathcal{K}_f$
- Forgetful functor $\mathbf{Law}(\mathcal{K}) \to [|\mathcal{K}_f|^2, \mathcal{V}]$ sending $\mathcal{L}$ to

$$\big(\mathcal{L}(Ec, Ed)\big)_{c,d \in |\mathcal{K}_f|}$$

is finitarily monadic, and so $\mathbf{Law}(\mathcal{K})$ is LFP and is the category of models of a Lawvere theory in $[|\mathcal{K}_f|^2, \mathcal{V}]$.
(cf Lack theorem on monadicity of $\mathbf{Mnd}_\mathrm{f}(\mathcal{K})$ over $[|\mathcal{K}_f|, \mathcal{K}]$.)

The **Cat**-enriched case

- Take $\mathcal{V} = \mathbf{Cat}$.
- Finitary 2-monads on an LFP 2-category $\mathcal{K}$ are equivalent to Lawvere 2-theories $E : \mathcal{K}_f^{\mathrm{op}} \to \mathcal{L}$
- Can describe such structures as monoidal category, category with limits and/or colimits of some type, categories with limits and colimits and exactness conditions, category with two monoidal structures and a distributive law, category with a factorization system, pair of monoidal categories with a monoidal adjunction etc.
- Need to weaken this to get lax and pseudo versions for example, want functors which preserve limits in the usual sense, not on-the-nose; and want monoidal or strong monoidal functors, not strict ones

Pseudomorphisms

- For 2-category $\mathcal{C}$, write $\mathrm{Ps}(\mathcal{C}, \mathbf{Cat})$ for the 2-category of (strict) 2-functors, pseudonatural transformations, and modifications and $[\mathcal{C}, \mathbf{Cat}]$ for sub-2-category of strict maps
- If $\mathcal{K}$ LFP have maps $\mathcal{K} \to [\mathcal{K}_f^{\mathrm{op}}, \mathbf{Cat}] \to \mathrm{Ps}(\mathcal{K}_f^{\mathrm{op}}, \mathbf{Cat})$
- For theory $\mathcal{L}$ define 2-category $\mathrm{Mod}(\mathcal{L})_{\mathrm{ps}}$ of strict models and pseudomaps by pullback

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{L})_{\mathrm{ps}} & \longrightarrow & \mathrm{Ps}(\mathcal{L}, \mathbf{Cat}) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Ps}(E, \mathbf{Cat})} \\ \mathcal{K} & \longrightarrow & \mathrm{Ps}(\mathcal{K}_f^{\mathrm{op}}, \mathbf{Cat}) \end{array}$$

- So a pseudomap between models $X, X' : \mathcal{L} \to \mathbf{Cat}$ is a pseudonatural transformation $f : X \rightsquigarrow X'$ with fE strict:

$$\mathcal{K}_f^{\mathrm{op}} \xrightarrow{E} \mathcal{L} \underset{X'}{\overset{X}{\rightrightarrows}} \mathbf{Cat} \quad (\Downarrow f)$$

More on pseudomorphisms

- for a model X, we have $XC = (X1)^C$ where $X1$ is the underlying object
- $\mathcal{L}(C,1)$ is the category of C-ary operations
- for $\gamma : C \to 1$ in $\mathcal{L}$, have operation $X\gamma : (X1)^C \to X1$
- for a pseudomorphism $f : X \to Y$ have pseudonaturality isomorphisms

$$\begin{array}{ccc} (X1)^C & \xrightarrow{X\gamma} & X1 \\ {\scriptstyle (f1)^C}\downarrow & \cong & \downarrow{\scriptstyle f1} \\ (Y1)^C & \xrightarrow[Y\gamma]{} & Y1 \end{array}$$

which show how f preserves the operation γ up to isomorphism

Comparision with 2-dimensional monad theory

- Given a finitary 2-monad T on $\mathcal{K}$, we can consider the 2-category T-**Alg** of strict T-algebras and pseudo T-morphisms [cf Blackwell-Kelly-Power]
- Alternatively we can consider the corresponding theory $\mathcal{L}$, and $\text{Mod}(\mathcal{L})_{\text{ps}}$ as above
- The equivalence $\text{Mod}(\mathcal{L}) \simeq \mathcal{K}^T$ extends to an equivalence $\text{Mod}(\mathcal{L})_{\text{ps}} \simeq T\text{-}\mathbf{Alg}$.
- Analogous results for lax morphisms

Weakening the models

- Write $\mathrm{Hom}(\mathcal{C}, \mathbf{Cat})$ for the 2-category of pseudofunctors (homomorphisms), pseudonatural transformations, and modifications from $\mathcal{C}$ to $\mathbf{Cat}$
- Define 2-category of pseudomodels and pseudomorphisms by the pullback

$$\begin{array}{ccc} \text{Ps-Mod}(\mathcal{L}) & \longrightarrow & \mathrm{Hom}(\mathcal{L}, \mathbf{Cat}) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(E, \mathbf{Cat})} \\ \mathcal{K} & \longrightarrow & \mathrm{Hom}(\mathcal{K}_f^{\mathrm{op}}, \mathbf{Cat}) \end{array}$$

- A pseudomodel is a pseudofunctor $\mathcal{L} \to \mathbf{Cat}$ whose restriction to $\mathcal{K}_f^{\mathrm{op}}$ is strict and preserves finite limits.
- Once again this agrees with the monad-theoretic notion
- Once again there are lax versions

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