

# JÓNSSON - TARSKI TOPOSES

or

## The Space of an Endomodule

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## Terminology

Let  $\mathcal{A}$  and  $\mathcal{B}$  be small categories.

$$\hat{\mathcal{A}} = [\mathcal{A}^{\text{op}}, \text{Set}] \quad (= \text{"right } \mathcal{A}\text{-modules"})$$

An  $(\mathcal{A}, \mathcal{B})$ -module  $M$  is a functor  $M: \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}$ .

Then  $M$  induces an adjunction

$$\hat{\mathcal{A}} \begin{array}{c} \xrightarrow{- \otimes M} \\ \perp \\ \xleftarrow{[M, -]} \end{array} \hat{\mathcal{B}}$$

where, for  $\gamma \in \hat{\mathcal{B}}$  and  $a \in \mathcal{A}$ ,

$$[M, \gamma](a) = \text{Hom}(M(-, a), \gamma).$$

A  $\text{two-sided } \mathcal{A}\text{-module}$  (or  $\mathcal{A}\text{-endomodule}$ )

is an  $(\mathcal{A}, \mathcal{A})$ -module.

# "The space of an endomodule"

Any two-sided module  
gives rise to  
a topological object

in at least two ways.

## ① Self-similarity

Roughly,  $M$  gives rise to

a functor  $I_M: \mathcal{A} \rightarrow \text{Top}$ ,

the terminal coalgebra for  $[\mathcal{A}, \text{Top}] \hookrightarrow \mathbf{M}_0$ .

$\mathcal{A}$ : category  
 $M$ :  $(\mathcal{A}, \mathcal{A})$ -module

## ② This talk

$M$  gives rise to

a topos  $\mathcal{J}T_M$ ,

the "Jónsson-Tarski topos of  $M$ " (using  $\hat{\mathcal{A}} \hookrightarrow [\mathbf{M}, -]$ ).

## The classical Jónsson-Tarski topos (1961.)

A Jónsson-Tarski algebra  $(X, \xi)$  is a set  $X$  equipped with a bijection

$$\xi : X \xrightarrow{\sim} X^2 = X \times X$$

$$\mathcal{JT}_2 = \text{Fix} \left( \text{Set}^{(\cdot)^2} \right)$$

They form a category  $\mathcal{JT}_2$ .

Three non-obvious things:

1. Jónsson-Tarski algebras are an algebraic theory
2. (free algebra on  $X$ )  $\cong$  (free algebra on  $X \times 2$ )  
for any set  $X$  (and in particular,  $F1 \cong F2$ )
3.  $\mathcal{JT}_2$  is a topos (Freyd).

# The classical Jónsson-Tarski topos (cont)

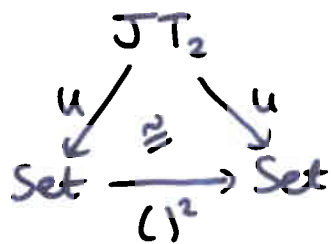
## Proofs:

1. A Jónsson-Tarski algebra is a set  $X$  with operations

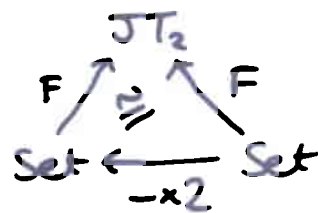
$$l, r: X \rightarrow X, \quad \cdot: X^2 \rightarrow X$$

satisfying certain equations.

2.



$\mapsto$



3. Site is free monoid on 2 generators  $\lambda, \rho$ ;

$\{\lambda, \rho\}$  is a cover.

## General Jónsson-Tarski toposes

Let  $\mathcal{A}$  be a small category and  $M$  an  $(\mathcal{A}, \mathcal{A})$ -module.

A Jónsson-Tarski  $M$ -algebra  $(X, \xi)$  is a presheaf  $X \in \hat{\mathcal{A}}$  equipped with an isomorphism

$$\xi: X \xrightarrow{\sim} [M, X].$$

$$\mathcal{J}T_M = \text{Fix} \left( \hat{\mathcal{A}} \xrightarrow{[M, -]} \hat{\mathcal{A}} \right)$$

They form a category  $\mathcal{J}T_M$ .

E.g.:  $\mathcal{A} = \mathbf{1}$ ,  $M = 2$ : then  $\mathcal{J}T_M = \mathcal{J}T_2$ .

E.g.: Any  $\mathcal{A}$ ,  $M = \text{Hom}$ : then  $\mathcal{J}T_M = [\mathbb{Z}, \hat{\mathcal{A}}] = \widehat{\mathbb{Z} \times \mathcal{A}}$ .

Three non-obvious things:

1.  $\mathcal{J}T_M$  is monadic over  $\hat{\mathcal{A}}$
2. (free algebra on  $X$ )  $\cong$  (free algebra on  $X \otimes M$ )  
for any  $X \in \hat{\mathcal{A}}$
3.  $\mathcal{J}T_M$  is a topos.

## General Jónsson-Tarski toposes (cont.)

Proof of 3:

Define a site  $\mathcal{A}_M$  by adjoining to  $\mathcal{A}$  one new arrow  $b \xrightarrow{m} a$  for each  $b, a \in \mathcal{A}$  and  $m \in M(b, a)$ .

For each  $a$ , the family of such arrows covers  $a$ .

Then  $\mathcal{J}\mathcal{T}_M = \text{Sh}(\mathcal{A}_M)$ .

## Finite discrete case

Take  $I = \{1, \dots, n\}$ , discrete cat, and  $M: I^{\text{op}} \times I \rightarrow \text{FinSet}$ .

Then  $M$  is an  $n \times n$  matrix  $(\mu_{ij})$  of natural numbers.

A Jónsson-Tarski  $M$ -algebra consists of sets

$X_1, \dots, X_n$  together with bijections

$$\xi_1: X_1 \xrightarrow{\sim} X_1^{\mu_{11}} \times \dots \times X_n^{\mu_{1n}}$$

⋮

$$\xi_n: X_n \xrightarrow{\sim} X_1^{\mu_{n1}} \times \dots \times X_n^{\mu_{nn}}$$

Regard  $M$  as a directed graph with vertices  $1, \dots, n$ .

Then the site  $\mathcal{A}_M$  is the free category on this graph.



## Example: "real interval"

Let  $A = (0 \rightrightarrows 1)$ ; then  $\hat{A} = \text{DirGph}$ .

There is an  $(A, A)$ -module  $M$  such that if

$X = (X_1 \rightrightarrows X_0) \in \hat{A}$  then

$$[M, X] = (X_1, x_{x_0} X_1 \rightrightarrows X_0).$$

A Jónsson-Tarski  $M$ -algebra is a graph  $X$

with an isomorphism

$$\xi: \begin{pmatrix} X_1 \\ \Downarrow \\ X_0 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} X_1, x_{x_0} X_1 \\ \Downarrow \\ X_0 \end{pmatrix}.$$

(When  $\xi_0 = 1_{X_0}$ , this is a "bijective composition" on  $X$ .)

E.g.: The "1-skeleton" of a space is a

Jónsson-Tarski  $M$ -algebra:

$$\begin{array}{ccc} & & \text{JT}_M \\ & \dashrightarrow & \downarrow u \\ \text{Top} & \xrightarrow{\pi_1} & \hat{A} = \text{DirGph} \end{array}$$

## Open questions

- Which toposes are Jónsson-Tarski?

**Thm:** Every presheaf topos is JT

**Thm:**  $\text{Sh}(X)$  is JT for every compact totally disconnected metric space  $X$

**Thm (Lacke):** If  $\mathcal{E}$  is JT and  $E \in \mathcal{E}$  then  $\mathcal{E}/E$  is JT.

(Is every Grothendieck topos Jónsson-Tarski?)

- A two-sided module  $M$  gives rise to two topological objects, the topos  $\text{JT}_M$  and the functor  $I_M: \mathbb{A} \rightarrow \text{Top}$  ( $\approx$  terminal coalgebra for  $[\mathbb{A}, \text{Top}] \stackrel{M \otimes -}{\curvearrowright}$ ). How are they related?

**E.g.:** Classical case ( $\mathbb{A} = \mathbf{1}$ ,  $M = 2$ ): have

$\text{JT}_M = \text{JT}_2$ ,  $I_M = 2^{\mathbb{N}}$  (Cantor set), and

$\text{JT}_2 / F(1) \cong \text{Sh}(2^{\mathbb{N}})$ .