JÓNSSON-TARSKI TOPOSES

or

The Space of an Endomodule

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Terminology

Let $\mathbb{A}$ and $\mathbb{B}$ be small categories.

$\hat{\mathbb{A}} = [\mathbb{A}^{op}, \mathbf{Set}]$ (= "right $\mathbb{A}$-modules")

An $(\mathbb{A}, \mathbb{B})$-module M is a functor $M : \mathbb{B}^{op} \times \mathbb{A} \longrightarrow \mathbf{Set}$.

Then M induces an adjunction

$$\begin{tikzcd} \hat{\mathbb{A}} \arrow[r, shift left, "- \otimes M"] \arrow[r, phantom, "\perp"] & \hat{\mathbb{B}} \arrow[l, shift left, "{[M, -]}"] \end{tikzcd}$$

where, for $Y \in \hat{\mathbb{B}}$ and $a \in \mathbb{A}$,

$$[M, Y](a) = \mathrm{Hom}(M(-, a), Y).$$

A two-sided $\mathbb{A}$-module (or $\mathbb{A}$-endomodule) is an $(\mathbb{A}, \mathbb{A})$-module.

"The space of an endomodule"

Any two-sided module gives rise to a topological object

in at least two ways.

① Self-similarity

Roughly, M gives rise to

a functor $I_M : \mathbb{A} \longrightarrow \mathrm{Top}$,

the terminal coalgebra for $[\mathbb{A}, \mathrm{Top}] \circlearrowleft M \otimes -$.

$\mathbb{A}$: category
M: $(\mathbb{A},\mathbb{A})$-module

② This talk

M gives rise to

a topos $\mathcal{JT}_M$,

the "Jónsson-Tarski topos of M" (using $\hat{\mathbb{A}} \circlearrowleft [M,-]$).

The classical Jónsson-Tarski topos
(1961)

A Jónsson-Tarski algebra (X, ξ) is a set X equipped with a bijection

$$\xi : X \xrightarrow{\sim} X^2 = X \times X$$

$$JT_2 = \mathrm{Fix}\left(\mathbf{Set} \circlearrowleft (\,)^2\right)$$

They form a category JT_2.

Three non-obvious things:

1. Jónsson-Tarski algebras are an algebraic theory
2. (free algebra on X) $\cong$ (free algebra on $X \times 2$) for any set X (and in particular, $F1 \cong F2$)
3. JT_2 is a topos (Freyd).

The classical Jónsson-Tarski topos (cont)

Proofs:

1. A Jónsson-Tarski algebra is a set X with operations

$$l, r : X \longrightarrow X, \qquad \bullet : X^2 \longrightarrow X$$

satisfying certain equations.

2.
$$\begin{tikzcd} & JT_2 \arrow[dl, "u"'] \arrow[dr, "u"] & \\ Set \arrow[rr, "(\ )^2"'] \arrow[rr, phantom, bend left=20, "\cong"] & & Set \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & JT_2 & \\ Set \arrow[ur, "F"] \arrow[rr, phantom, bend left=20, "\cong"] & & Set \arrow[ul, "F"'] \arrow[ll, "-\times 2"] \end{tikzcd}$$

$$\begin{tikzcd} JT_2 \arrow[d, bend left, "u"] \arrow[d, phantom, "\dashv"] \\ Set \arrow[u, bend left, "F"] \end{tikzcd}$$

3. Site is free monoid on 2 generators λ, ρ; $\{\lambda, \rho\}$ is a cover.

General Jónsson-Tarski toposes

Let $\mathbb{A}$ be a small category and M an $(\mathbb{A},\mathbb{A})$-module.

A Jónsson-Tarski M-algebra (X, ξ) is a presheaf $X \in \hat{\mathbb{A}}$ equipped with an isomorphism

$$\xi : X \xrightarrow{\sim} [M, X].$$

$\rightsquigarrow \mathcal{JT}_M = \mathrm{Fix}\left(\hat{\mathbb{A}} \circlearrowleft^{[M,-]} \right)$

They form a category $\mathcal{JT}_M$.

E.g.: $\mathbb{A} = \mathbb{1}$, $M = 2$: then $\mathcal{JT}_M = \mathcal{JT}_2$.

E.g.: Any $\mathbb{A}$, $M = \mathrm{Hom}$: then $\mathcal{JT}_M = [\mathbb{Z}, \hat{\mathbb{A}}] = \widehat{\mathbb{Z} \times \mathbb{A}}$.

Three non-obvious things:

1. $\mathcal{JT}_M$ is monadic over $\hat{\mathbb{A}}$
2. (free algebra on X) $\cong$ (free algebra on $X \otimes M$) for any $X \in \hat{\mathbb{A}}$
3. $\mathcal{JT}_M$ is a topos.

General Jónsson-Tarski toposes (cont.)

Proof of 3:

Define a site $\mathbb{A}_M$ by adjoining to $\mathbb{A}$ one new arrow $b \overset{m}{\nrightarrow} a$ for each $b, a \in \mathbb{A}$ and $m \in M(b,a)$.

For each a, the family of such arrows covers a.

Then $JT_M = Sh(\mathbb{A}_M)$.

Finite discrete case

Take $\mathbb{A} = \{1, \dots, n\}$, discrete cat, and $M : \mathbb{A}^{op} \times \mathbb{A} \to \mathrm{FinSet}$.
Then M is an $n \times n$ matrix (μ_{ij}) of natural numbers.

A Jónsson-Tarski M-algebra consists of sets
$X_1, \dots, X_n$ together with bijections

$$\xi_1 : X_1 \xrightarrow{\sim} X_1^{\mu_{11}} \times \dots \times X_n^{\mu_{1n}}$$
$$\vdots$$
$$\xi_n : X_n \xrightarrow{\sim} X_1^{\mu_{n1}} \times \dots \times X_n^{\mu_{nn}}.$$

Regard M as a directed graph with vertices $1, \dots, n$.
Then the site $\mathbb{A}_M$ is the free category on this graph.

Example: "real interval"

Let $\mathbb{A} = (0 \rightrightarrows 1)$; then $\hat{\mathbb{A}} = \mathrm{DirGph}$.

There is an $(\mathbb{A},\mathbb{A})$-module M such that if

$X = (X_1 \rightrightarrows X_0) \in \hat{\mathbb{A}}$ then

$$[M,X] = (X_1 \times_{X_0} X_1 \rightrightarrows X_0).$$

A Jónsson-Tarski M-algebra is a graph X with an isomorphism

$$\xi : \begin{pmatrix} X_1 \\ \downdownarrows \\ X_0 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} X_1 \times_{X_0} X_1 \\ \downdownarrows \\ X_0 \end{pmatrix}.$$

(When $\xi_0 = 1_{X_0}$, this is a "bijective composition" on X.)

E.g.: The "1-skeleton" of a space is a Jónsson-Tarski M-algebra:

$$\begin{tikzcd} & JT_M \arrow[d, "u"] \\ \mathrm{Top} \arrow[ur, dashed] \arrow[r, "\pi_1"'] & \hat{\mathbb{A}} = \mathrm{DirGph} \end{tikzcd}$$

Open questions

- Which toposes are Jónsson–Tarski?

  Thm: Every presheaf topos is JT

  Thm: $Sh(X)$ is JT for every compact totally disconnected metric space X

  Thm (Lack): If $\mathcal{E}$ is JT and $E \in \mathcal{E}$ then $\mathcal{E}/E$ is JT.

(Is <u>every</u> Grothendieck topos Jónsson–Tarski?)

- A two-sided module M gives rise to <u>two</u> topological objects, the topos JT_M and the functor $I_M : \mathbb{A} \longrightarrow Top$ ($\approx$ terminal coalgebra for $[\mathbb{A}, Top] \circlearrowleft^{M \otimes -}$). How are they related?

  E.g.: Classical case ($\mathbb{A} = \mathbb{1}$, $M = 2$): have $JT_M = JT_2$, $I_M = 2^{\mathbb{N}}$ (Cantor set), and

  $$JT_2 / F(1) \simeq Sh(2^{\mathbb{N}}).$$