# Semistrict Tamsamani's n-groupoids and connected n-types


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## Overview

- Modelling homotopy $n$-types:

| Homotopy Theory | Higher categories |
|---|---|
| Cat$^{n-1}$-groups (path-connected case) | Tamsamani weak $n$-groupoids |

- Semistrictification hypothesis
  Weak $n$-groupoids suitably equivalent to "semistrict" ones. Simpson's conjecture.

- Low dimensions
  $n = 2$ strict 2-groupoids model 2-types.
  $n = 3$ - Gray groupoids model 3-types [Joyal -Tierney; Leroy]
  - One object 3-groupoids with weak units model 1-connected 3-types [Joyal - Kock]

- Main result. Every Tamsamani weak $n$-groupoid representing a connected $n$-type is equivalent to a "semistrict" one via zig-zag of $n$-equivalences.

- Method. Comparison between cat$^{n-1}$-groups and Tamsamani's weak $n$-groupoids.

**$\mathrm{Cat}^n$-groups as homotopy models.**

- Definition: $\mathrm{Cat}^0(\mathrm{Gp}) = \mathrm{Gp}$
  $\mathrm{Cat}^n(\mathrm{Gp}) = \mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathrm{Gp}))$

- Classifying space $B$
  $\mathrm{Cat}^n(\mathrm{Gp}) \xrightarrow{\mathcal{N}} [\Delta^{n^{op}}, \mathrm{Gp}] \xrightarrow{diag} [\Delta^{op}, \mathrm{Gp}] \xrightarrow{\overline{W}}$
  $[\Delta^{op}, \mathbf{Set}]_0 \xrightarrow{|\cdot|} \mathrm{Top}_*$

- Fact: $\mathcal{G} \in \mathrm{Cat}^n(\mathrm{Gp})$
  Then $B\mathcal{G}$ is connected $(n+1)$-type.

- Weak equivalence $f$ in $\mathrm{Cat}^n(\mathrm{Gp})$ if $Bf$ weak homotopy equivalence.

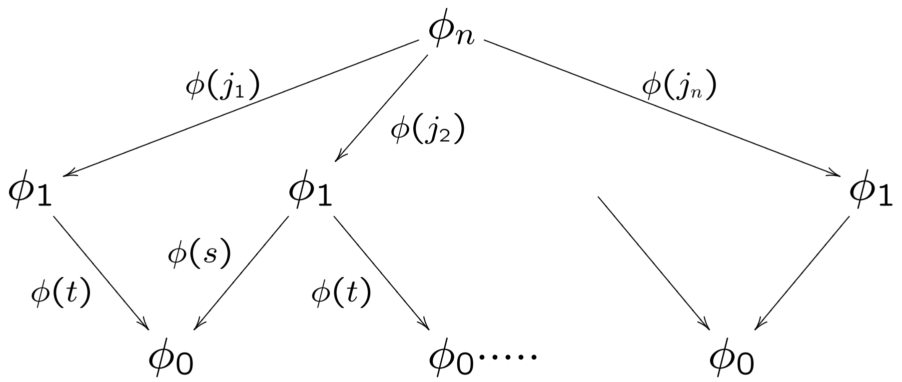
- Theorem

[MacLane-Whitehead $n = 1$]
[Loday; Bullejos-Cegarra-Duskin;
Porter; $n > 1$]

$$\overline{B} : \frac{\mathrm{Cat}^n(\mathrm{Gp})}{\sim} \simeq \mathcal{H}o\begin{pmatrix}\text{connected}\\ n+1\text{-types}\end{pmatrix} : \overline{\mathcal{P}}_n$$

## Segal maps.

- Segal maps $\phi \in [\Delta^{op}, \mathcal{C}]$ $\quad \phi[n] = \phi_n$




$t(0) = 1 \quad s(0) = 0 \quad j_r(0) = r - 1 \quad j_r(1) = r$

Hence maps

$$\eta_n : \phi_n \to \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1$$

- Fact: $\phi$ nerve of Cat$\,\mathcal{C}$ iff Segal maps isomorphisms

- Fact: For suitable $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}\text{-Cat} &\simeq \{\psi \in \text{Cat}\,\mathcal{C} \mid \psi_0 \text{ discrete}\} \\ &\cong \{\phi \in [\Delta^{op}, \mathcal{C}] \mid \phi_0 \text{ discrete, Segal maps isos}\} \end{aligned}$$

**Segal maps: Examples**

a) Inductive definition of $\text{Cat}^n$-groups.

$n = 1$ Cat (Gp)

Given $\text{Cat}^{n-1}$(Gp)

$\phi \in \text{Cat}^n(\text{Gp})$ if $\phi \in [\Delta^{op}, \text{Cat}^{n-1}(\text{Gp})]$

with Segal maps isomorphisms.

b) Inductive definition of strict $n$-categories.

$n = 1$ Cat

Given $(n-1)$-Cat

$\phi \in n$-Cat if $\phi \in [\Delta^{op}, (n-1)\text{-Cat}]$ with

(i) $\phi_0$ discrete

(ii) Segal maps isomorphisms.

- Note: In ex. b), $\mathcal{NG}(0,\text{-})$, $\mathcal{NG}(m_1 \overset{k}{\ldots} m_k\, 0, \text{-})$ $1 \leq k \leq n-2$ are constant (globularity condition). Not the case in general in example a).

## Tamsamani's model

- Idea: weaken associativity of composition and unit laws by requiring Segal maps to be "equivalences" rather than isomorphisms.

- Inductive definition [Tamsamani; Toen]:

  $\mathcal{W}_1 = \mathsf{Cat}$, 1-equivalence = equiv. of categories
  $\tau_0^{(1)} : \mathsf{Cat} \to \mathbf{Set}$ iso class of objects
  $\delta^{(1)} : \mathbf{Set} \to \mathcal{W}_1$ discrete category
  $\tau_1^{(1)} = \mathrm{id} : \mathcal{W}_1 \to \mathsf{Cat}$

  Inductive step:

- Given $\mathcal{W}_{n-1}$, $(n-1)$-equivalences
  $\tau_0^{(n-1)} : \mathcal{W}_{n-1} \to \mathbf{Set}$
  $\delta^{(n-1)} : \mathbf{Set} \to \mathcal{W}_{n-1}$ image "discrete"
  + axioms

- Define $\phi \in \mathcal{W}_n \subset [\Delta^{op}, \mathcal{W}_{n-1}]$ s.t.
  $\phi_0$ discrete
  Segal maps $(n-1)$-equivalences.

- Note $\tau_1^{(n)} : \mathcal{W}_n \to \mathsf{Cat}$ restriction of
  $\overline{\tau}_0^{(n-1)} : [\Delta^{op}, \mathcal{W}_{n-1}] \to [\Delta^{op}, \mathbf{Set}]$
  $\phi_1 \cong \coprod_{x,y \in \phi_0} \phi_{(x,y)}$

- Define $f : \phi \to \psi$ in $\mathcal{W}_n$ $n$-equivalence if
  - $\phi_{(x,y)} \to \psi_{(fx,fy)}$ $(n-1)$-equiv.
  - $\tau_1^{(n)}\phi \to \tau_1^{(n)}\psi$ equiv. of Cat
  - $\tau_0^{(n)} = \tau_0^{(1)}\tau_1^{(n)}$

**Weak $n$-groupoids as homotopy models.**

- Tamsamani's weak $n$-groupoids $\mathcal{T}_n \subset \mathcal{W}_n$

  $\underline{n = 1}$ $\mathcal{T}_1 = \text{Gpd}$

  Given $\mathcal{T}_{(n-1)}$

  Define $\phi \in \mathcal{T}_n \subset \mathcal{W}_n$ if

  $\phi_{(x,y)} \in \mathcal{T}_{n-1}$ for all $x, y \in \phi_0$

  $\tau_1^{(n)} \phi \in \mathcal{T}_1$

- Note: $\mathcal{N} : \mathcal{T}_n \to [\Delta^{n^{op}}, \mathbf{Set}]$

- Theorem [Tamsamani]

  Equivalence of categories

$$\overline{B} : \frac{\mathcal{T}_n}{\sim^n} \simeq \mathcal{H}o\,(n\text{-types}) : \overline{\mathbf{\Pi}}_n$$

## Comparison problem: overview.

- Comparison method:

$$\text{Cat}^n(\text{Gp}) \xrightarrow{disc} \mathbb{D}_n \xrightarrow{V} \mathcal{H}_{n+1}$$

| strict cubical structure internal to Gp | weak globular structure internal to Gp | semistrict Tamsamani $(n+1)$-groupoids |
|---|---|---|

- $\phi \in \mathcal{H}_{n+1} \subset \mathcal{T}_{n+1} \subset [\Delta^{op}, \mathcal{T}_n]$ iff
  $\phi_0 = \{*\}$, $\phi_n \cong \phi_1 \times_{\phi_o} \overset{n}{\cdots} \times_{\phi_0} \phi_1$
- $\mathbb{D}_n \subset [\Delta^{n^{op}}, \text{Gp}]$ internal weak $n$-groupoids.
- $V$ induced by nerve $\text{Gp} \to [\Delta^{op}, \mathbf{Set}]$
- $disc$ and $V$ preserve the homotopy type.

- Discretization functor: is composite

$$\text{Cat}^n(\text{Gp}) \xrightarrow{Sp} \text{Cat}^n(\text{Gp})_s \xrightarrow{\mathcal{D}_n} \mathbb{D}_n$$

- $\text{Cat}^n(\text{Gp})_s \subset \text{Cat}^n(\text{Gp})$ special cat$^n$-groups. The "faces" which in $n$-Cat (Gp) are discrete are now "strongly contractible".

- $Sp$ composite of functorial cofibrant replacements.

- $\mathcal{D}_n$ "squeezes contractible faces to discrete ones".

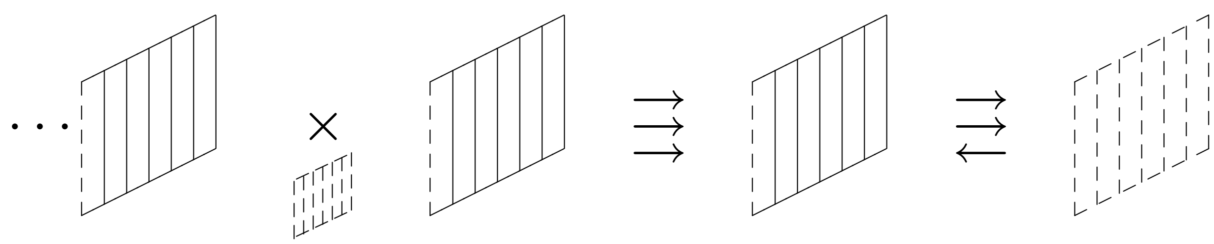
**Special $\mathrm{cat^n}$-groups.**

- Notation $\mathcal{G} \in \mathsf{Cat}^{\,n}(\mathsf{Gp})$
  $\mathcal{N}\mathcal{G}(x_1 \cdots x_{k-1}\ i\ x_{k+1} \cdots x_n)$ multinerves of $\mathsf{Cat}^{\,n-1}(\mathsf{Gp})$ denoted by $\mathcal{G}_i^{(k)}$, $i = 0, 1$.

- Strongly contractible (s.c.) $\mathrm{cat}^n$-groups
  $n = 1$ $\quad d : \mathcal{G} \rightleftarrows \mathcal{G}^d : t \quad dt = \mathrm{id}$
  $\mathcal{G}^d$ discrete, $\quad d$ w.e.
  Given s.c. $\mathsf{Cat}^{\,n-1}(\mathsf{Gp})$
  Define $\mathcal{G} \in \mathsf{Cat}^{\,n}(\mathsf{Gp})$ s.c. if
  $d : \mathcal{G} \rightleftarrows \mathcal{G}^d : t \quad dt = \mathrm{id}$
  $\mathcal{G}^d$ discrete, $\quad d$ w.e.
  $\mathcal{G}_0^{(k)} \quad \mathcal{G}_1^{(k)}$ s.c. $\mathsf{Cat}^{\,n-1}(\mathsf{Gp})$
  for all directions $1 \le k \le n$.

- Special $\mathsf{Cat}^{\,n}(\mathsf{Gp})$, inductive definition.
  $\mathcal{G} \in \mathsf{Cat}^{\,n}(\mathsf{Gp})_s$ if $\mathcal{G}_0^{(k)}$ is s.c. and $\mathcal{G}_1^{(k)}$ is special for some direction $k$, $1 \le k \le n$.

- Example: $\mathcal{G} \in \mathsf{Cat}^{\,3}(\mathsf{Gp})_s$, $\mathcal{N}\mathcal{G}$ looks like:



special $\mathsf{Cat}^{\,2}(\mathsf{Gp})$ s.c. $\mathsf{Cat}^{\,2}(\mathsf{Gp})$

## The discretization functor.

- Internal weak n-groupoids $\mathbb{D}_1 = \mathsf{Cat}\,(\mathsf{Gp})$,
  $n > 1$ $\mathcal{G} \in \mathbb{D}_n \subset [\Delta^{op}, \mathbb{D}_{n-1}]$ with $\mathcal{G}_0$ discrete, Segal maps w.e.

- Discrete multinerve

  $\mathcal{G} \in \mathsf{Cat}^{\,n}(\mathsf{Gp})_s \quad d : \mathcal{G}_0 \rightleftarrows \mathcal{G}_0^d : t \quad dt = \mathrm{id}$

  $$Ner\,\mathcal{G} \qquad \cdots \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathcal{G}_1 \;\substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xleftarrow[\sigma_0]{}}\; \mathcal{G}_0$$

  $$ds\,\mathcal{N}\mathcal{G} \qquad \cdots \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathcal{G}_1 \;\substack{\xrightarrow{d\partial_0}\\ \xrightarrow{d\partial_1}\\ \xleftarrow[\sigma_0 t]{}}\; \mathcal{G}_0^d$$

  hence $ds\,\mathcal{N} : \mathsf{Cat}^{\,n}(\mathsf{Gp})_s \to [\Delta^{op}, \mathsf{Cat}^{\,n-1}(\mathsf{Gp})_s]$

- Functor $\mathcal{D}_2 = ds\,\mathcal{N} : \mathsf{Cat}^{\,2}(\mathsf{Gp})_s \to \mathbb{D}_2$
  $n > 2$, $\mathcal{D}_n : \mathsf{Cat}^{\,n}(\mathsf{Gp})_s \to \mathbb{D}_n$
  $\mathcal{D}_n = \overline{\mathcal{D}}_{n-1} \circ ds\,\mathcal{N}$
  preserves homotopy type

- Discretization $disc : \mathsf{Cat}^{\,n}(\mathsf{Gp}) \to \mathbb{D}_n$
  $disc = \mathcal{D}_n \circ Sp$

**Semistrictification, general n.**

- Semistrict Tamsamani's $n+1$-groupoids $\mathcal{H}_{n+1}$
  $\phi \in \mathcal{H}_{n+1} \subset \mathcal{T}_{n+1} \subset [\Delta^{op}, \mathcal{T}_n\,]$ if
  $\phi_0 = \{*\}, \quad \phi_n \cong \phi_1 \times \overset{n}{\cdots} \times \phi_1$

- Functor $V : \mathbb{D}_n \to \mathcal{H}_{n+1}$
  induced by nerve $\mathsf{Gp} \to [\Delta^{op}, \mathbf{Set}]$
  $V$ preserves homotopy type.

- Theorem [P.] Commutative diagram

$$
\begin{array}{ccc}
\mathsf{Cat}^{\,n}(\mathsf{Gp})/\!\sim & \xrightarrow{V \circ disc} & \mathcal{H}_{n+1}/\!\sim^{n+1} \\
& {\scriptstyle B}\searrow \quad \swarrow{\scriptstyle B} & \\
& \mathcal{H}o\begin{pmatrix}\text{connected} \\ n+1\text{-types}\end{pmatrix} &
\end{array}
$$

Further, every object of $\mathcal{T}_{n+1}$ representing a connected $(n+1)$-type is equivalent to an object of $\mathcal{H}_{n+1}$ through a zig-zag of $(n+1)$-equivalences in $\mathcal{T}_{n+1}$.