Quantal sets, quantale modules, and groupoid actions

PEDRO RESENDE

(Joint work with M. Protin and E. Rodrigues)

Instituto Superior Técnico
Lisboa, Portugal
Motivations

- Sheaves on spaces or locales $\sim$ local structure (e.g., scheme, smooth manifold, etc.)
Motivations

- Sheaves on spaces or locales $\sim$ local structure (e.g., scheme, smooth manifold, etc.)
- Examples of non-local structure on a space $X$:
  - Equivalence relation on $X$
  - Group action on $X$
  - $X = G_0$ for a topological groupoid $G$
Motivations

- Sheaves on spaces or locales \( \sim \) local structure (e.g., scheme, smooth manifold, etc.)

- Examples of non-local structure on a space \( X \):
  - Equivalence relation on \( X \)
  - Group action on \( X \)
  - \( X = G_0 \) for a topological groupoid \( G \)

- Sheaves on \( X \) need to be compatible with the non-local structure (e.g., equivariant sheaves on a groupoid)
Motivations

Sheaves on spaces or locales $\sim$ local structure (e.g., scheme, smooth manifold, etc.)

Examples of non-local structure on a space $X$:
- Equivalence relation on $X$
- Group action on $X$
- $X = G_0$ for a topological groupoid $G$

Sheaves on $X$ need to be compatible with the non-local structure (e.g., equivariant sheaves on a groupoid)

Non-equivalent alternative: sheaves on quotient space $X/G$; or on the orbit space of a groupoid; or on the primitive spectrum of a C*-algebra, etc.
Motivations

- Quantales generalize locales
- Several notions of “sheaf” on them: Borceux, van den Bossche, Gyllys, Mulvey, Nawaz, van der Plancke, Stubbe, Walters, Zamora Ramos, etc.
Motivations

- Quantales generalize locales
- Several notions of “sheaf” on them: Borceux, van den Bossche, Gylys, Mulvey, Nawaz, van der Plancke, Stubbe, Walters, Zamora Ramos, etc.
- Quantales generalize (open) groupoids:
  étale groupoids = inverse quantal frames
Motivations

- Quantales generalize locales
- Several notions of “sheaf” on them: Borceux, van den Bossche, Gyllys, Mulvey, Nawaz, van der Plancke, Stubbe, Walters, Zamora Ramos, etc.
- Quantales generalize (open) groupoids:
  étale groupoids = inverse quantal frames
- Question: how do equivariant sheaves of groupoids translate to the quantale language?
Past talks:

equivariant sheaves on $G \sim \text{“certain modules”}$
over quantale $\mathcal{O}(G)$
Past talks:

equivariant sheaves on $G \sim \text{“certain modules”}$
over quantale $\mathcal{O}(G)$

This talk:

- replace \textit{“certain modules”} by a definition
Past talks:
equivariant sheaves on \( G \sim \text{“certain modules”} \)
over quantale \( \mathcal{O}(G) \)

This talk:
- replace \text{“certain modules”} by a definition
- definition should be simple!
Past talks:

equivariant sheaves on $G \sim \text{"certain modules"}$
over quantale $\mathcal{O}(G)$

This talk:

- replace “certain modules” by a definition
- definition should be simple!
- sheaf theory as “linear algebra”: matrices over
locales à-la Fourman and Scott (1979); Hilbert
$\mathbb{Q}$-modules as in Paseka’s work
Past talks:

equivariant sheaves on $G \sim \text{“certain modules”}$
over quantale $\mathcal{O}(G)$

This talk:

replace “certain modules” by a definition

definition should be simple!

sheaf theory as “linear algebra”: matrices over locales à-la Fourman and Scott (1979); Hilbert $Q$-modules as in Paseka’s work

For an étale groupoid $G$ (at least if $G_0$ is, say, T1):

classifying topos $BG \sim \text{category of (…) - } \mathcal{O}(G)$-modules

Ok for the étale groupoids arising in geometry and analysis
Notation and terminology

If $X$ is a locale we refer to $\mathcal{O}(X)$ as itself in the dual category $Frm = Loc^{op}$ of frames.

A *map of locales* $p : X \to B$ is defined by its *inverse image* frame homomorphism $p^* : \mathcal{O}(B) \to \mathcal{O}(X)$ in $Frm$.

Then $\mathcal{O}(X)$ is an $\mathcal{O}(B)$-module by “change of ring” along $p^*$; the action is given by, for $b \in \mathcal{O}(B)$ and $x \in \mathcal{O}(X)$,

$$bx = p^*(b) \land x.$$  

Hence, $b1 = p^*(b)$, and thus the action satisfies $bx = b1 \land x$.

A frame $\mathcal{O}(X)$ equipped with such a module structure is called an $\mathcal{O}(B)$-*locale*. 
Notation and terminology

The category $\mathcal{O}(B)$-$\text{Loc}$ of $\mathcal{O}(B)$-locales is the opposite of the category whose objects are the $\mathcal{O}(B)$-locales and whose arrows are the $\mathcal{O}(B)$-equivariant frame homomorphisms.

Recall that $p : X \to B$ is open if $p^*$ has a left adjoint $p_!$ (the direct image of $p$) which is $\mathcal{O}(B)$-equivariant. Then we say that $\mathcal{O}(X)$ is an open $\mathcal{O}(B)$-locale.

If $p$ is a local homeomorphism then $\mathcal{O}(X)$ is an étale $\mathcal{O}(B)$-locale.
Theorem. $\text{Loc}/B \simeq \mathcal{O}(B)$-Loc
Theorem. \( \text{Loc}/B \simeq \mathcal{O}(B)\text{-Loc} \)

Theorem. If \( \mathcal{O}(X) \) has a monotone equivariant map

\[ \varsigma : \mathcal{O}(X) \to \mathcal{O}(B) \]

such that \( (\varsigma x)x = x \) then \( \mathcal{O}(X) \) is open.
**Theorem.** \( 	ext{Loc}/B \simeq \mathcal{O}(B)-\text{Loc} \)

**Theorem.** If \( \mathcal{O}(X) \) has a monotone equivariant map

\[ \varsigma : \mathcal{O}(X) \to \mathcal{O}(B) \]

such that \( (\varsigma x)x = x \) then \( \mathcal{O}(X) \) is open.

**Definition.** Let \( \mathcal{O}(X) \) be an open \( B \)-locale. A *local section* of \( \mathcal{O}(X) \) is an element \( s \in \mathcal{O}(X) \) such that for all \( x \leq s \) we have \( (\varsigma x)s = x \). The set of local sections is \( \Gamma_X \).

**Theorem.** An open \( \mathcal{O}(B) \)-locale \( \mathcal{O}(X) \) is étale if and only if \( \bigvee \Gamma_X = 1 \).
Local sections as “basis vectors”

\[ x = x \wedge 1 = x \wedge \bigvee \Gamma_X = \bigvee_{s \in \Gamma_X} x \wedge s = \bigvee_{s \in \Gamma_X} \varsigma(x \wedge s)s = \bigvee_{s \in \Gamma_X} \langle x, s \rangle s \]

where \( \langle x, y \rangle \), defined to be \( \varsigma(x \wedge y) \), is a symmetric “bilinear” form

\[ \langle - , - \rangle : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(B) \]

with “Hilbert basis” \( \Gamma_X \).
Theorem. \( \mathcal{O}(X) \) is an étale \( \mathcal{O}(B) \)-locale if and only if there is a symmetric bilinear form (the “inner product”)

\[ \langle - , - \rangle : \mathcal{O}(X) \times \mathcal{O}(X) \rightarrow \mathcal{O}(B) \]

and a subset \( \Gamma \subset X \) (the “Hilbert basis”) such that for all \( x \in \mathcal{O}(X) \) we have

\[ x = \bigvee_{s \in \Gamma} \langle x, s \rangle s \]

The homomorphisms of such modules are adjointable (for any homomorphism \( f : \mathcal{O}(X) \rightarrow \mathcal{O}(Y) \) there is a unique \( f^\dagger : \mathcal{O}(Y) \rightarrow \mathcal{O}(X) \) such that \( \langle f(x), y \rangle = \langle x, f^\dagger(y) \rangle \)) and the resulting category \( B\text{-HB} \) has two subcategories, resp. isomorphic to \( LH/B \) and to \( (LH/B)^{op} \).
Theorem. $B$-HB is equivalent to the category whose objects are the projection matrices over $\mathcal{O}(B)$,

$$E : \Gamma \times \Gamma \rightarrow \mathcal{O}(B)$$

$$E = E^2 = E^T$$

and whose morphisms $T : E \rightarrow F$ are the matrices such that

$$ET = T = TF$$
Groupoids

\[
\begin{array}{c}
G_2 \xrightarrow{m} G_1 \\
\pi_2 \xrightarrow{\pi_1} G_1 \\
G_1 \xrightarrow{r} G_0
\end{array}
\]

\[
\begin{array}{c}
G_2 \xrightarrow{\pi_2} G_1 \\
\pi_1 \\
G_1 \xrightarrow{r} G_0
\end{array}
\]

\[
\begin{array}{c}
G_1 \xrightarrow{\pi_1} G_0
\end{array}
\]
Groupoids

\[
G_2 \xrightarrow{m} G_1 \xleftarrow{u} G_0 \\
G_2 \xrightarrow{\pi_2} G_1 \xrightarrow{\pi_1} G_1 \\
G_2 \xrightarrow{\pi_1} G_1 \xrightarrow{m} G_0 \\
G_1 \xrightarrow{r} G_0 \xrightarrow{d} G_0
\]
Groupoids

\[ G_2 \xrightarrow{m} G_1 \xrightarrow{r} G_0 \]

\[ G_2 \xrightarrow{\pi_2} G_1 \]
\[ G_1 \xrightarrow{r} G_0 \]

\[ d \text{ open} \Rightarrow m \text{ open} \]

\[ d \text{ local homeomorphism} \Rightarrow m \text{ local homeomorphism} \]
Étale groupoids

\[ \mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \rightarrow \mathcal{O}(G_2) \xrightarrow{m!} \mathcal{O}(G_1) \]

\[ \mathcal{O}(G_1) \xrightarrow{i_1} \mathcal{O}(G_1) \]

\( \mathcal{O}(G_1) \) is an involutive quantale. It is unital if and only if \( G \) is étale:

\[ e = G_0 \in \mathcal{O}(G_1) \]

We denote this quantale by \( \mathcal{O}(G) \).
**Involutive quantales**

**Definition.** A *unital involutive quantale* $Q$ is an involutive monoid,

\[(ab)c = a(bc)\]

\[ae = a\]

\[ea = a\]

\[a^{**} = a\]

\[(ab)^* = b^*a^* ,\]

in the monoidal category of sup-lattices:

\[(\bigvee a_i)b = \bigvee a_ib\]

\[b(\bigvee a_i) = \bigvee ba_i\]

\[(\bigvee a_i)^* = \bigvee a_i^* .\]

**Notation.**  
\[1 = \bigvee Q \quad 0 = \bigvee \emptyset\]
Groupoid quantales

Theorem. (R 2007) Let $\mathcal{Q}$ be an inverse quantal frame, i.e., a unital involutive quantale that is also a locale satisfying simple properties, in particular

$$1 = \bigvee \mathcal{I}(\mathcal{Q})$$

where

$$\mathcal{I}(\mathcal{Q}) = \{ a \in \mathcal{Q} \mid aa^* \leq e, \ a^*a \leq e \} \quad (\text{"partial units" of } \mathcal{Q})$$

(= inverse semigroup)

Then $\mathcal{Q} \cong \mathcal{O}(G)$ for an étale groupoid $G$. 
Groupoid quantales

From an inverse quantal frame $Q$ with multiplication

\[ \mu : Q \otimes \downarrow(e) \rightarrow Q \]

define

\[ G(Q) = \begin{array}{c}
G_2 \\
\circlearrowleft
\end{array} G_1 \begin{array}{c}
m \\
\rightarrow
\end{array} \begin{array}{c}
G_0
\end{array} \]

\[ O(G_1) = Q \quad O(G_0) = \downarrow(e) \]
Groupoid quantales

From an inverse quantal frame $Q$ with multiplication

$$\mu : Q \otimes \downarrow(e) Q \to Q$$

define

$$\mathcal{G}(Q) = G_2 \xrightarrow{m} G_1 \xleftarrow{u} G_0$$

$$\mathcal{O}(G_1) = Q \quad \quad \quad \mathcal{O}(G_0) = \downarrow(e)$$

$$m^* = \mu^* \quad \text{(Tricky!)}$$
Groupoid actions

\[
\begin{array}{c}
G_1 \\ \downarrow d \\
G_0 \\
\end{array}
\quad \text{acts on} \quad \begin{array}{c}
X \\ \downarrow p \\
G_0 \\
\end{array}
\]

\[
\begin{array}{c}
G_1 \times_{r,p} X \\ \downarrow \pi_1 \\
G_1 \\
\end{array} \quad \alpha \quad \begin{array}{c}
X \\ \downarrow p \\
G_0 \\
\end{array}
\]

pullback
Groupoid actions

\[ \begin{array}{ccc}
G_1 & \text{acts on} & X \\
\downarrow d & & \downarrow p \\
G_0 & & G_0 \\
\end{array} \]

\[ G_1 \times_{r,p} X \xrightarrow{\alpha} X \]

\[ \begin{array}{ccc}
G_1 & \xrightarrow{d} & G_0 \\
\downarrow \pi_1 & & \downarrow p \\
G_1 & & G_0 \\
\end{array} \]

\[ d \text{ open } \Rightarrow \alpha \text{ open} \]
Groupoid actions as modules

\( \mathcal{O}(X) \) is a left \( \mathcal{O}(G) \)-module:

\[
\mathcal{O}(G) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(G_1 \times_{r,p} X) \overset{\alpha_!}{\longrightarrow} \mathcal{O}(X)
\]
Groupoid actions as modules

\( \mathcal{O}(X) \) is a left \( \mathcal{O}(G) \)-module:

\[
\mathcal{O}(G) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(G_1 \times_{r,p} X) \alpha! \longrightarrow \mathcal{O}(X)
\]

The assignment \( X \mapsto \mathcal{O}(X) \) is functorial (due to Beck–Chevalley): we obtain a faithful functor

\[
\mathcal{B}G \rightarrow (\mathcal{O}(G)\text{-}\mathbf{Mod})^{\text{op}}
\]
Groupoid actions as modules

\( \mathcal{O}(X) \) is a left \( \mathcal{O}(G) \)-module:

\[
\mathcal{O}(G) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(G_1 \times_{r,p} X) \alpha! \longrightarrow \mathcal{O}(X)
\]

The assignment \( X \mapsto \mathcal{O}(X) \) is functorial (due to Beck–Chevalley): we obtain a faithful functor

\[
\mathcal{B}G \rightarrow (\mathcal{O}(G)\text{-Mod})^{\text{op}}
\]

The functor is also full for some groupoids (for instance if \( G_0 \) is a T1 space).
Definition. Let $Q$ be an involutive quantale. A Hilbert $Q$-module with a Hilbert basis is a left $Q$-module $M$ equipped with a $Q$-valued “inner product”

$$\langle - , - \rangle : M \times M \rightarrow Q$$

$$\bigvee S, y = \bigvee_{x \in S} \langle x, y \rangle$$

$$\langle ax, y \rangle = a \langle x, y \rangle$$

$$\langle y, x \rangle = \langle x, y \rangle^*$$

and a subset $\Gamma \subseteq M$ such that for all $x \in M$ we have

$$x = \bigvee_{s \in \Gamma} \langle x, s \rangle s$$
Properties

Parseval’s formula: $\langle x, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle$. 
Parseval’s formula: \( \langle x, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle \).

Restriction of \( \langle -, - \rangle \) to \( \Gamma \times \Gamma \) is a projection matrix \( E : \Gamma \times \Gamma \to Q \):

\[
E = E^* = E^2
\]
Properties

Parseval’s formula: \( \langle x, y \rangle = \bigvee_{s \in \Gamma} \langle x, s \rangle \langle s, y \rangle \).

Restriction of \( \langle -, - \rangle \) to \( \Gamma \times \Gamma \) is a projection matrix \( E : \Gamma \times \Gamma \to Q \):

\[ E = E^* = E^2 \]

From such a matrix construct Hilbert \( Q \)-module \( M \):

\[ M = Q^\Gamma E \]

The Hilbert basis of \( M \) can be identified with the set of rows of \( E \).

Equivalence of categories!
Now $G$ is étale and $X \to G_0$ is a l.h.

Hence, $\mathcal{O}(X)$ is an étale $\mathcal{O}(G_0)$-locale, and the action of $G$ on $X$ restricts to an action

$$\mathcal{I}(\mathcal{O}(G)) \times \Gamma_X \to \Gamma_X.$$
Groupoid sheaves as Hilbert modules

Now $G$ is étale and $X \to G_0$ is a l.h.

Hence, $\mathcal{O}(X)$ is an étale $\mathcal{O}(G_0)$-locale, and the action of $G$ on $X$ restricts to an action

$$\mathcal{I}(\mathcal{O}(G)) \times \Gamma_X \to \Gamma_X.$$

Define matrix $A : \Gamma_X \times \Gamma_X \to \mathcal{O}(G)$ as follows:

For each pair $s, t \in \Gamma_X$ let

$$a_{st} = \bigvee \{ f \in \mathcal{I}(\mathcal{O}(G)) \mid ff^* \leq \varsigma s, \ f^*f \leq \varsigma t, \ ft \leq s \}$$
Hilbert modules as groupoid sheaves

$Q$: inverse quantal frame of the étale groupoid $G$

$M$: A Hilbert étale $Q$-locale, i.e., a locale which is also a Hilbert $Q$-module with basis $\Gamma$ such that

$$\bigvee \Gamma = 1$$
Hilbert modules as groupoid sheaves

$Q$: inverse quantal frame of the étale groupoid $G$

$M$: A *Hilbert étale $Q$-locale*, i.e., a locale which is also a Hilbert $Q$-module with basis $Γ$ such that

$$\bigvee Γ = 1$$

**Theorem.** There is a (unique up to iso) l.h. $X → G_0$ with a $G$-action such that

$$M \cong \mathcal{O}(X)$$

as left $Q$-modules.
Proof sketch

Step 1: show that $\mathcal{M}$ is an étale $\downarrow(e)$-locale with inner product $\langle x, y \rangle_\ell = \langle x, y \rangle \wedge e$
Proof sketch

Step 1: show that $M$ is an étale $\downarrow(e)$-locale with inner product $\langle x, y \rangle_\ell = \langle x, y \rangle \wedge e$

Step 2: show that action $Q \otimes M \to M$ restricts to $\mathcal{I}(Q) \times \Gamma \to \Gamma_X$
Proof sketch

Step 1: show that $M$ is an étale $\downarrow(e)$-locale with inner product $\langle x, y \rangle_\ell = \langle x, y \rangle \land e$

Step 2: show that action $Q \otimes M \to M$ restricts to $\mathcal{I}(Q) \times \Gamma \to \Gamma_X$

Step 3: show that previous action extends to action on the join-completions of $\mathcal{I}(Q)$ and $\Gamma_X$:

$$\alpha : \mathcal{L}^\land(\mathcal{I}(Q)) \times \downarrow(e) \mathcal{L}^\land(\Gamma_X) \to \mathcal{L}^\land(\Gamma_X)$$
Proof sketch

Step 1: show that $M$ is an étale $\downarrow(e)$-locale with inner product $\langle x, y \rangle_\ell = \langle x, y \rangle \wedge e$

Step 2: show that action $Q \otimes M \rightarrow M$ restricts to $I(Q) \times \Gamma \rightarrow \Gamma_X$

Step 3: show that previous action extends to action on the join-completions of $I(Q)$ and $\Gamma_X$:

$$a : \mathcal{L}^\vee(I(Q)) \times_\downarrow(e) \mathcal{L}^\vee(\Gamma_X) \rightarrow \mathcal{L}^\vee(\Gamma_X)$$

Step 4: we know from [R 2007] that $\mathcal{L}^\vee(I(Q)) \cong Q$ and a similar argument shows that $\mathcal{L}^\vee(\Gamma_X) \cong M$. 
Proof sketch

Step 1: show that $M$ is an étale $\downarrow(e)$-locale with inner product $\langle x, y \rangle_\ell = \langle x, y \rangle \wedge e$

Step 2: show that action $Q \otimes M \rightarrow M$ restricts to $\mathcal{I}(Q) \times \Gamma \rightarrow \Gamma_X$

Step 3: show that previous action extends to action on the join-completions of $\mathcal{I}(Q)$ and $\Gamma_X$:

$$\alpha : \mathcal{L}^\vee(\mathcal{I}(Q)) \times \downarrow(e) \mathcal{L}^\vee(\Gamma_X) \rightarrow \mathcal{L}^\vee(\Gamma_X)$$

Step 4: we know from [R 2007] that $\mathcal{L}^\vee(\mathcal{I}(Q)) \cong Q$ and a similar argument shows that $\mathcal{L}^\vee(\Gamma_X) \cong M$.

Step 5: show that $\alpha_*$ preserves joins: proof similar to that of multiplicativity in [R 2007]
Final remarks

- Étale is great!
Final remarks

- Étale is great!
- Full and faithful functor $\mathcal{B}G \to (\mathcal{O}(G)\text{-Hilb}_{\text{EtaleLoc}})^{\text{op}}$; equivalence for many groupoids
Final remarks

- Étale is great!
- Full and faithful functor $BG \rightarrow (\mathcal{O}(G)\text{-HilbEtaleLoc})^{\text{op}}$; equivalence for many groupoids
- Quantale homomorphisms versus geometric morphisms of toposes...
Final remarks

- Étale is great!
- Full and faithful functor $\mathcal{B}G \to (\mathcal{O}(G)\text{-HilbEtaleLoc})^\text{op}$; equivalence for many groupoids
- Quantale homomorphisms versus geometric morphisms of toposes...
- Abelian sheaves... (→ cohomology, smooth quantales)
Final remarks

- Étale is great!
- Full and faithful functor $\mathcal{B}G \to (\mathcal{O}(G)\text{-HilbEtaleLoc})^{\text{op}}$; equivalence for many groupoids
- Quantale homomorphisms versus geometric morphisms of toposes...
- Abelian sheaves... (→ cohomology, smooth quantales)
- General comment: in [Joyal & Tierney 1984] locales are the “commutative algebra” of topos theory; but the complete ring-theoretic analogy requires more general quantales:
  Non-local geometry ∼ non-commutative (and non-idempotent) algebra