Quantal sets, quantale modules, and groupoid actions

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Motivations

- Sheaves on spaces or locales $\sim$ *local* structure (e.g., scheme, smooth manifold, etc.)

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- Examples of *non-local* structure on a space X:
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  - $X = G_0$ for a topological groupoid G
- Sheaves on X need to be compatible with the non-local structure (e.g., equivariant sheaves on a groupoid)
- Non-equivalent alternative: sheaves on quotient space X/G; or on the orbit space of a groupoid; or on the primitive spectrum of a C*-algebra, etc.

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- Quantales generalize locales
- Several notions of “sheaf” on them: Borceux, van den Bossche, Gylys, Mulvey, Nawaz, van der Plancke, Stubbe, Walters, Zamora Ramos, etc.

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- Quantales generalize (open) groupoids:

  étale groupoids = inverse quantal frames
- Question: how do equivariant sheaves of groupoids translate to the quantale language?

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- Past talks:

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  - sheaf theory as "linear algebra": matrices over locales à-la Fourman and Scott (1979); Hilbert Q-modules as in Paseka's work
- For an étale groupoid G (at least if G_0 is, say, T1): classifying topos $\mathcal{B}G \sim$ category of (...)-$\mathcal{O}(G)$-modules
- Ok for the étale groupoids arising in geometry and analysis

Notation and terminology

If X is a locale we refer to $\mathcal{O}(X)$ as itself in the dual category *Frm* $=$ *Loc*$^{\mathrm{op}}$ of frames.

A *map of locales* $p : X \to B$ is defined by its *inverse image* frame homomorphism $p^* : \mathcal{O}(B) \to \mathcal{O}(X)$ in *Frm*.

Then $\mathcal{O}(X)$ is an $\mathcal{O}(B)$-module by "change of ring" along p^*; the action is given by, for $b \in \mathcal{O}(B)$ and $x \in \mathcal{O}(X)$,

$$bx = p^*(b) \wedge x \ .$$

Hence, $b1 = p^*(b)$, and thus the action satisfies $bx = b1 \wedge x$.

A frame $\mathcal{O}(X)$ equipped with such a module structure is called an *$\mathcal{O}(B)$-locale*.

Notation and terminology

The category *$\mathcal{O}(B)$-Loc* of $\mathcal{O}(B)$-locales is the opposite of the category whose objects are the $\mathcal{O}(B)$-locales and whose arrows are the $\mathcal{O}(B)$-equivariant frame homomorphisms.

Recall that $p : X \longrightarrow B$ is *open* if p^* has a left adjoint $p_!$ (the *direct image* of p) which is $\mathcal{O}(B)$-equivariant. Then we say that $\mathcal{O}(X)$ is an *open* $\mathcal{O}(B)$-locale.

If p is a local homeomorphism then $\mathcal{O}(X)$ is an *étale* $\mathcal{O}(B)$-locale.

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Theorem. *If* $\mathcal{O}(X)$ *has a monotone equivariant map*

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such that $(\varsigma x)x = x$ *then* $\mathcal{O}(X)$ *is open.*

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Definition. Let $\mathcal{O}(X)$ be an open B-locale. A *local section* of $\mathcal{O}(X)$ is an element $s \in \mathcal{O}(X)$ such that for all $x \leq s$ we have $(\varsigma x)s = x$. The set of local sections is $\mathit{\Gamma}_X$.

Theorem. *An open* $\mathcal{O}(B)$*-locale* $\mathcal{O}(X)$ *is étale if and only if* $\bigvee \mathit{\Gamma}_X = 1$*.*

Local sections as “basis vectors”

$$x = x \wedge 1 = x \wedge \bigvee \Gamma_X = \bigvee_{s \in \Gamma_X} x \wedge s = \bigvee_{s \in \Gamma_X} \varsigma(x \wedge s) s = \bigvee_{s \in \Gamma_X} \langle x, s \rangle s$$

where $\langle x, y \rangle$, defined to be $\varsigma(x \wedge y)$, is a symmetric “bilinear” form

$$\langle -, - \rangle : \mathcal{O}(X) \times \mathcal{O}(X) \to \mathcal{O}(B)$$

with “Hilbert basis” Γ_X.

Theorem. *$\mathcal{O}(X)$ is an étale $\mathcal{O}(B)$-locale if and only if there is a symmetric bilinear form (the " inner product")*

$$\langle -,-\rangle : \mathcal{O}(X)\times\mathcal{O}(X)\to\mathcal{O}(B)$$

and a subset $\Gamma\subset X$ (the "Hilbert basis") such that for all $x\in\mathcal{O}(X)$ we have

$$x=\bigvee_{s\in\Gamma}\langle x,s\rangle s$$

The homomorphisms of such modules are adjointable (for any homomorphism $f:\mathcal{O}(X)\to\mathcal{O}(Y)$ there is a unique $f^{\dagger}:\mathcal{O}(Y)\to\mathcal{O}(X)$ such that $\langle f(x),y\rangle=\langle x,f^{\dagger}(y)\rangle$) and the resulting category B-HB has two subcategories, resp. isomorphic to LH$/B$ *and to* $($LH$/B)^{op}$.

Theorem. *B-HB is equivalent to the category whose objects are the projection matrices over $\mathcal{O}(B)$,*

$$E : \Gamma \times \Gamma \longrightarrow \mathcal{O}(B)$$

$$E = E^2 = E^T$$

and whose morphisms $T : E \to F$ are the matrices such that

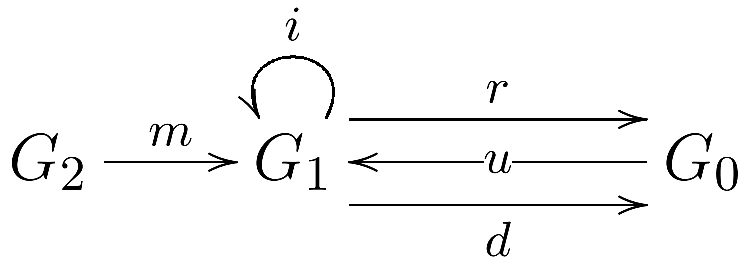
$$ET = T = TF$$

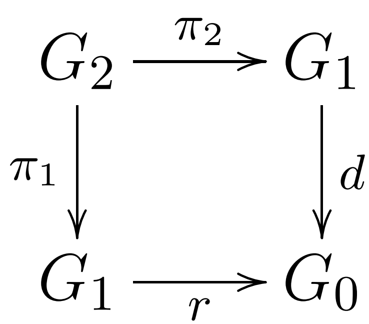
Groupoids

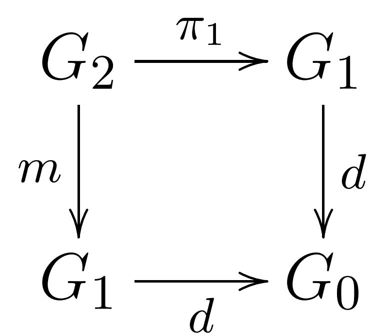
$$G_2 \xrightarrow{\;m\;} G_1 \circlearrowleft^{i} \quad \begin{array}{c} \xrightarrow{\quad r \quad} \\ \xleftarrow{\quad u \quad} \\ \xrightarrow[\quad d \quad]{} \end{array} \quad G_0$$

$$\begin{array}{ccc} G_2 & \xrightarrow{\;\pi_2\;} & G_1 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ G_1 & \xrightarrow[\;r\;]{} & G_0 \end{array}$$

Groupoids







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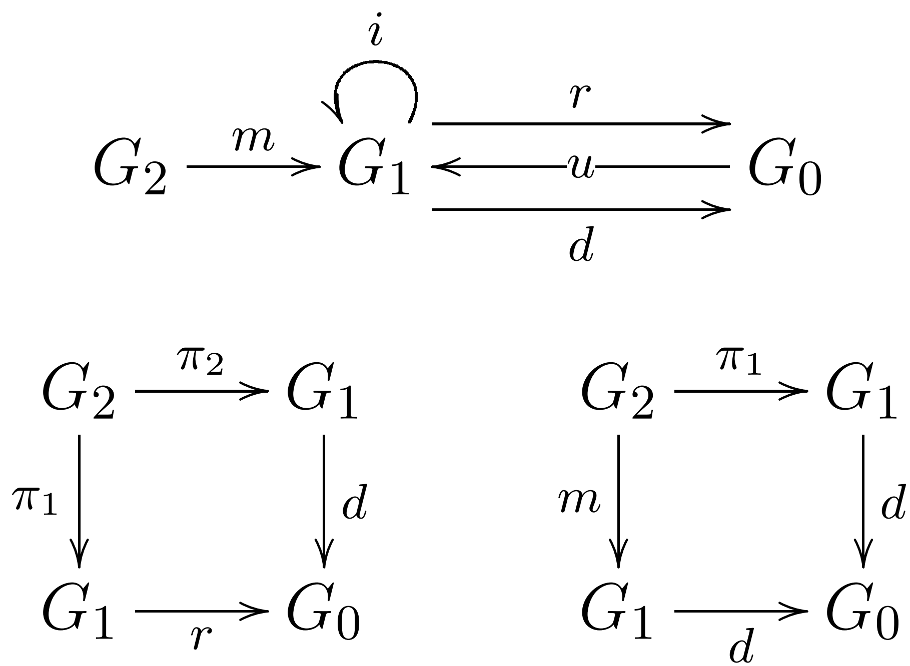

d open $\Rightarrow$ m open

d local homeomorphism $\Rightarrow$ m local homeomorphism

Étale groupoids

$$\mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \twoheadrightarrow \mathcal{O}(G_2) \xrightarrow{m_!} \mathcal{O}(G_1)$$

$$\mathcal{O}(G_1) \xrightarrow[\cong]{i_!} \mathcal{O}(G_1)$$

$\mathcal{O}(G_1)$ is an involutive quantale. It is unital if and only if G is étale:

$$e = G_0 \in \mathcal{O}(G_1)$$

We denote this quantale by $\mathcal{O}(G)$.

Involutive quantales

Definition. A *unital involutive quantale* Q is an involutive monoid,

$$\begin{aligned} (ab)c &= a(bc) \\ ae &= a \\ ea &= a \\ a^{**} &= a \\ (ab)^* &= b^* a^* \ , \end{aligned}$$

in the monoidal category of sup-lattices:

$$\begin{aligned} (\bigvee a_i)b &= \bigvee a_i b \\ b(\bigvee a_i) &= \bigvee b a_i \\ (\bigvee a_i)^* &= \bigvee a_i^* \ . \end{aligned}$$

Notation. $1 = \bigvee Q \qquad 0 = \bigvee \emptyset$

Groupoid quantales

Theorem. (R 2007) *Let Q be an* inverse quantal frame*, i.e., a unital involutive quantale that is also a locale satisfying simple properties, in particular*

$$1 = \bigvee \mathcal{I}(Q)$$

where

$$\mathcal{I}(Q) \quad = \quad \{a \in Q \mid aa^* \leq e,\ a^*a \leq e\} \quad (\text{"partial units" of } Q)$$

(= inverse semigroup)

Then $Q \cong \mathcal{O}(G)$ for an étale groupoid G.

Groupoid quantales

From an inverse quantal frame Q with multiplication

$$\mu : Q \otimes_{\downarrow(e)} Q \to Q$$

define

$$\mathcal{G}(Q) \quad = \quad G_2 \xrightarrow{m} G_1 \overset{i}{\circlearrowleft} \underset{d}{\overset{r}{\rightrightarrows}} G_0, \qquad u : G_0 \to G_1$$

$$\mathcal{O}(G_1) \;=\; Q \qquad\qquad \mathcal{O}(G_0) \;=\; \downarrow(e)$$

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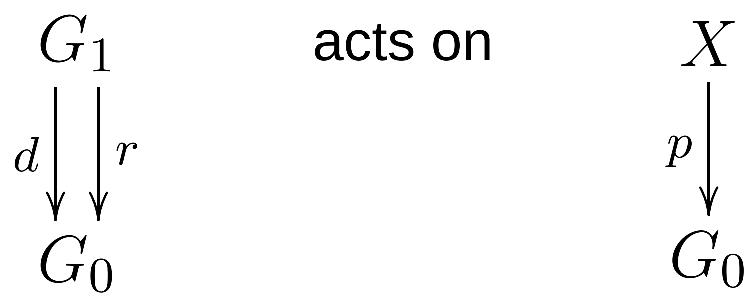
$$m^* \;=\; \mu_*$$ (Tricky!)

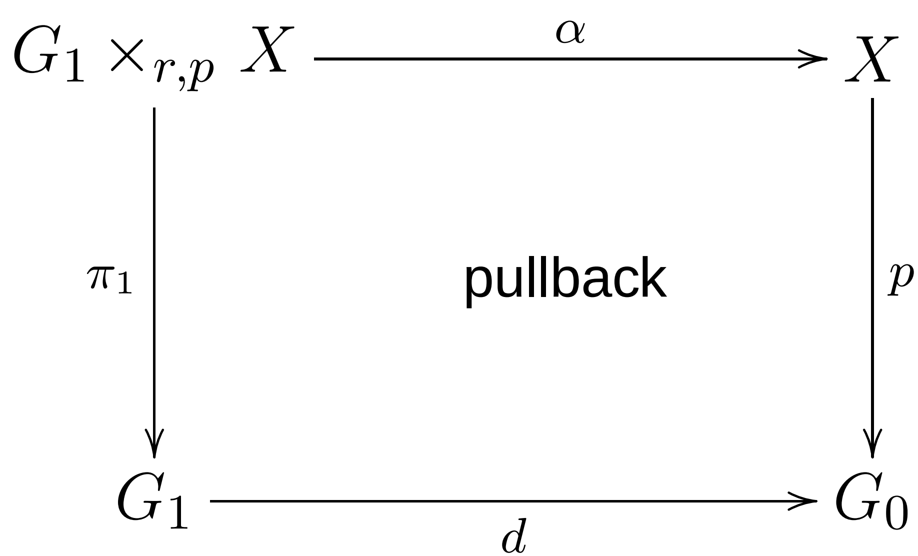
Groupoid actions

$$
\begin{array}{ccc}
G_1 & \text{acts on} & X \\
d \big\downarrow \big\downarrow r & & \big\downarrow p \\
G_0 & & G_0
\end{array}
$$

$$
\begin{array}{ccc}
G_1 \times_{r,p} X & \xrightarrow{\ \alpha\ } & X \\
\pi_1 \big\downarrow & \text{pullback} & \big\downarrow p \\
G_1 & \xrightarrow[\ d\ ]{} & G_0
\end{array}
$$

Groupoid actions





d open $\Rightarrow$ α open

Groupoid actions as modules

$\mathcal{O}(X)$ is a left $\mathcal{O}(G)$-module:

$$\mathcal{O}(G) \otimes \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(G_1 \times_{r,p} X) \xrightarrow{\alpha_!} \mathcal{O}(X)$$

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The functor is also full for some groupoids (for instance if G_0 is a T1 space).

Hilbert modules

Definition. Let Q be an involutive quantale. A *Hilbert Q-module* with a *Hilbert basis* is a left Q-module M equipped with a Q-valued "inner product"

$$\langle -, - \rangle : M \times M \to Q$$

$$\begin{aligned} \left\langle \bigvee S, y \right\rangle &= \bigvee_{x \in S} \langle x, y \rangle \\ \langle ax, y \rangle &= a \langle x, y \rangle \\ \langle y, x \rangle &= \langle x, y \rangle^* \end{aligned}$$

and a subset $\Gamma \subset M$ such that for all $x \in M$ we have

$$x = \bigvee_{s \in \Gamma} \langle x, s \rangle s$$

Properties

Parseval's formula: $\langle x, y\rangle = \bigvee_{s\in\Gamma}\langle x, s\rangle\langle s, y\rangle$.

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Restriction of $\langle -, -\rangle$ to $\Gamma \times \Gamma$ is a projection matrix $E : \Gamma \times \Gamma \to Q$:

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From such a matrix construct Hilbert Q-module M:

$$M \quad = \quad Q^{\Gamma} E$$

The Hilbert basis of M can be identified with the set of rows of E.

Equivalence of categories!

Groupoid sheaves as Hilbert modules

Now G is étale and $X \to G_0$ is a l.h.

Hence, $\mathcal{O}(X)$ is an étale $\mathcal{O}(G_0)$-locale, and the action of G on X restricts to an action

$$\mathcal{I}(\mathcal{O}(G)) \times \mathit{\Gamma}_X \to \mathit{\Gamma}_X \ .$$

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Define matrix $A : \Gamma_X \times \Gamma_X \to \mathcal{O}(G)$ as follows:

For each pair $s, t \in \Gamma_X$ let

$$a_{st} = \bigvee \{f \in \mathcal{I}(\mathcal{O}(G)) \mid ff^* \leq \varsigma s,\ f^*f \leq \varsigma t,\ ft \leq s\}$$

Hilbert modules as groupoid sheaves

Q: inverse quantal frame of the étale groupoid G

M: A *Hilbert étale Q-locale*, i.e., a locale which is also a Hilbert Q-module with basis Γ such that

$$\bigvee \Gamma = 1$$

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Theorem. *There is a (unique up to iso) l.h. $X \to G_0$ with a G-action such that*

$$M \cong \mathcal{O}(X)$$

as left Q-modules.

Proof sketch

Step 1: show that M is an étale $\downarrow(e)$-locale with inner product $\langle x, y\rangle_\ell = \langle x, y\rangle \wedge e$

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Step 3: show that previous action extends to action on the join-completions of $\mathcal{I}(Q)$ and $\mathit{\Gamma}_X$:

$$\mathfrak{a} : \mathcal{L}^\vee(\mathcal{I}(Q)) \times_{\downarrow(e)} \mathcal{L}^\vee(\mathit{\Gamma}_X) \to \mathcal{L}^\vee(\mathit{\Gamma}_X)$$

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Step 5: show that $\mathfrak{a}_*$ preserves joins: proof similar to that of multiplicativity in [R 2007]

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- General comment: in [Joyal & Tierney 1984] locales are the "commutative algebra" of topos theory; but the complete ring-theoretic analogy requires more general quantales:

  Non-local geometry $\sim$ non-commutative (and non-idempotent) algebra