Model Structures on the Category of Small Double Categories

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Overview

1. **Motivation**

2. Double Categories and Their Nerves

3. Model Categories

4. Results on Transfer from $[\Delta^{op}, \mathbf{Cat}]$ to $\mathbf{DblCat}$

5. Internal Point of View

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Motivation

When do we consider two categories A and B the same?
Two *different* possibilities:

1) If there is a functor $F : A \to B$ such that $NF : NA \to NB$ is a weak homotopy equivalence.

2) If there is a fully faithful and essentially surjective functor $F : A \to B$.

2) $\Rightarrow$ 1)

Motivation

Often one would like to invert the weak equivalences and have $Mor_{Ho\mathbf{C}}(D, E)$ be a set. Model structures enable one to do this.

Theorem 1 *(Thomason 1980) There is a model structure on* $\mathbf{Cat}$ *where* F *is a weak equivalence if and only if* NF *is a weak equivalence. Further, this model structure is Quillen equivalent to* $\mathbf{SSet}$*, and hence also* $\mathbf{Top}$*.*

Theorem 2 *(Joyal-Tierney 1991) There is a model structure on* $\mathbf{Cat}$ *where* F *is a weak equivalence if and only if* F *is an equivalence of categories.*

In this talk we consider similar questions for $\mathbf{DblCat}$. Since $\mathbf{DblCat}$ can be viewed in so many ways, there are many possible model structures.

Why are model structures on $\mathbf{DblCat}$ of interest?

1. Model categories have found great utility in the investigation of $(\infty, 1)$-categories.

Theorem 3 *(Bergner, Joyal-Tierney, Rezk,...) The following model categories are Quillen equivalent: simplicial categories, Segal categories, complete Segal spaces, and quasicategories.*

So we can expect them to also be of use in an investigation of iterated internalizations.

2. $\mathbf{DblCat}$ is useful for making sense of constructions in $\mathbf{Cat}$: calculus of mates, adjoining adjoints, spans (Dawson, Paré, Pronk, Grandis)

3. Parametrized Spectra (May-Sigurdsson, Shulman)

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Double Categories

Definition 1 *(Ehresmann 1963) A double category* $\mathbb{D}$ *is an internal category* $(\mathbb{D}_0, \mathbb{D}_1)$ *in* Cat.

Definition 2 *A small double category* $\mathbb{D}$ *consists of*
a set of objects,
a set of horizontal morphisms,
a set of vertical morphisms, and
a set of squares with source and target as follows

$$A \xrightarrow{f} B \qquad \begin{array}{c} A \\ {\scriptstyle j}\downarrow \\ C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

and compositions and units that satisfy the usual axioms and the interchange law.

Examples of Double Categories

1. Any 2-category is a double category with trivial vertical morphisms.

2. If $\mathbf{C}$ is a 2-category, then Ehresmann's double category of *quintets* $\mathbb{Q}\mathbf{C}$ has

$$Sq\,\mathbb{Q}\mathbf{C} := \left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & \alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \;\middle|\; A \underset{g\circ j}{\overset{k\circ f}{\underset{\Downarrow \alpha}{\rightrightarrows}}} D \right\}.$$

3. Rings, bimodules, ring maps, and twisted maps.

4. Categories, functors, profunctors, certain natural transformations.

Nerves of Double Categories

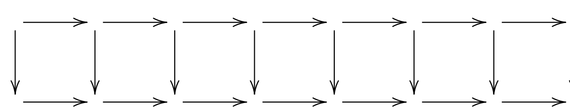
Horizontal Nerve:

$$N_h : \mathbf{DblCat} \to [\Delta^{op}, \mathbf{Cat}]$$

$$(N_h\mathbb{D})_n = \underbrace{(\mathbb{D}_1)\ {}_t\times_s\ (\mathbb{D}_1)\ {}_t\times_s \cdots {}_t\times_s\ (\mathbb{D}_1)}_{n \text{ copies}}$$

Obj : $\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow$

Mor :



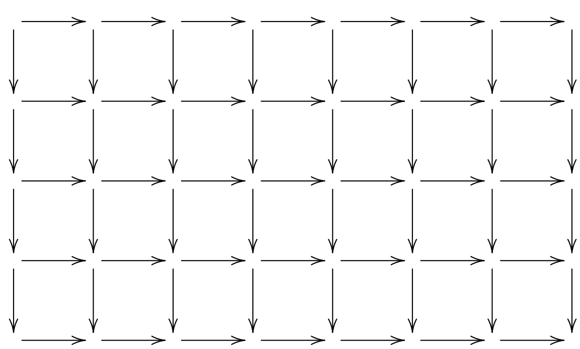
Proposition 4 *(FPP)* N_h *admits a left adjoint* c_h *called horizontal categorification.*

Nerves of Double Categories

Double Nerve:

$$N_d : \mathbf{DblCat} \to [\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}, \mathbf{Set}]$$

$$(N_d\mathbb{D})_{m,n} = \mathbf{DblCat}([m] \boxtimes [n], \mathbb{D})$$



Proposition 5 *(FPP) N_d admits a left adjoint c_d called double categorification.*

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Model Categories

A *model category* is a complete and cocomplete category $\mathbf{C}$ equipped with three subcategories:

1. weak equivalences
2. fibrations
3. cofibrations

which satisfy various axioms. Most notably: given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \text{cofibration } i \downarrow & & \downarrow p \text{ fibration} \\ B & \longrightarrow & Y \end{array}$$

in which at least one of i or p is a weak equivalence, then there exists a lift $h : B \longrightarrow X$.

Examples of Model Categories

It suffices to give weak equivalences and fibrations, since they determine together the cofibrations.

1. **Top** with π_*-isomorphisms and Serre fibrations.

2. **Cat** where F is a weak equivalence or fibration if and only if Ex^2NF is so (Thomason).

3. **Cat** with equivalences of categories and iso-fibrations (Joyal-Tierney).

4. $[\Delta^{op}, \mathbf{Cat}]$ with levelwise Thomason weak equivalences and levelwise Thomason fibrations.

5. $[\Delta^{op}, \mathbf{Cat}]$ with levelwise equivalences of categories and levelwise "iso-fibrations".

Model Structures on **2-Cat**

Theorem 6 *(Worytkiewicz, Hess, Parent, Tonks) There is a model structure on* **2-Cat** *in which a 2-functor* F *is a weak equivalence or fibration if and only if* $Ex^2 N_2 F$ *is.*

Theorem 7 *(Lack) There is a model structure on* **2-Cat** *in which the weak equivalences are 2-functors that are surjective on objects up to equivalence and locally an equivalence, and the fibrations are "equiv-fibrations".*

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Results on Transfer

Theorem 8 *(FPP) The levelwise Thomason model structure on* $[\Delta^{op}, \mathbf{Cat}]$ *transfers to a cofibrantly generated model structure on* $\mathbf{DblCat}$ *via horizontal categorification and horizontal nerve.*

$$[\Delta^{op}, \mathbf{Cat}] \underset{N_h}{\overset{c_h}{\rightleftarrows}} \mathbf{DblCat} \qquad \bot$$

$F : \mathbb{D} \longrightarrow \mathbb{E}$ *is a weak equivalence or fibration if and only if* $N_h F$ *is so.*

Results on Transfer

Theorem 9 *(FPP) The levelwise categorical model structure on* $[\Delta^{op}, \mathbf{Cat}]$ *transfers to a cofibrantly generated model structure on* $\mathbf{DblCat}$ *via horizontal categorification and horizontal nerve.*

$$[\Delta^{op}, \mathbf{Cat}] \underset{N_h}{\overset{c_h}{\rightleftarrows}} \mathbf{DblCat} \quad (\bot)$$

$F : \mathbb{D} \longrightarrow \mathbb{E}$ *is a weak equivalence or fibration if and only if* $N_h F$ *is so.*

Theorem 10 *(FPP) The Reedy categorical structure on* $[\Delta^{op}, \mathbf{Cat}]$ *cannot transfer to* $\mathbf{DblCat}$.

Main Technical Lemma

For the pushouts j_1 and j_2

$$\begin{array}{ccc} (cSd^2\Lambda^k[m]) \boxtimes [n] & \longrightarrow & \mathbb{D} \\ {\scriptstyle i\boxtimes 1_{[n]}}\big\downarrow & & \big\downarrow{\scriptstyle j_1} \\ (cSd^2\Delta[m]) \boxtimes [n] & \longrightarrow & \mathbb{P}_1 \end{array} \qquad \begin{array}{ccc} * \boxtimes [n] & \longrightarrow & \mathbb{D} \\ {\scriptstyle i\boxtimes 1_{[n]}}\big\downarrow & & \big\downarrow{\scriptstyle j_2} \\ I \boxtimes [n] & \longrightarrow & \mathbb{P}_2 \end{array}$$

in $\mathbf{DblCat}$ the morphisms $N_h(j_1)$ and $N_h(j_2)$ are weak equivalences in the respective model structures on $[\Delta^{op}, \mathbf{Cat}]$.

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Internal Point of View

Everaert, Kieboom, Van der Linden have shown that a Grothendieck topology on a good category $\mathcal{C}$ induces a model structure on $Cat(\mathcal{C})$ under certain hypotheses. We have two applications to $\mathcal{C} = \mathbf{Cat}$ so that $Cat(\mathcal{C}) = \mathbf{DblCat}$.

Grothendieck Topologies are of use because essential surjectivity does not make sense internally, but fully faithfullness does:

Definition 3 *(Bunge-Paré) An internal functor* $(F_0, F_1) : (A_0, A_1) \longrightarrow (B_0, B_1)$ *is fully faithful if*

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & B_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ A_0 \times A_0 & \xrightarrow[F_0 \times F_0]{} & B_0 \times B_0 \end{array}$$

is a pullback square in $\mathcal{C}$.

Essential $\mathcal{T}$-Surjectivity

Let $\mathcal{T}$ be a Grothendieck topology on **Cat**, and $\mathcal{E}_{\mathcal{T}}$ the class of functors p such that $Y_{\mathcal{T}}(p)$ is epi where $Y_{\mathcal{T}} : \mathbf{Cat} \to Sh(\mathbf{Cat}, \mathcal{T})$ is the composite of the Yoneda embedding with sheafification. $\mathcal{E}_{\mathcal{T}}$ is the class of *$\mathcal{T}$-epimorphisms*.

Definition 4 *A double functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is essentially $\mathcal{T}$-surjective if the functor*

$$(\mathbb{P}_F)_0 \longrightarrow \mathbb{B}_0$$

$$(a, f : b \xrightarrow{\cong} F_0 a) \mapsto b$$

is a $\mathcal{T}$-epimorphism.

Definition 5 *A $\mathcal{T}$-equivalence is a fully faithful double functor that is essentially $\mathcal{T}$-surjective.*

Model Structures on Categories of Internal Categories

Theorem 11 *(Everaert, Kieboom, Van der Linden)*

1. *Let $\mathcal{C}$ be a finitely complete category such that $Cat(\mathcal{C})$ is finitely complete and finitely cocomplete and $\mathcal{T}$ is a Grothendieck topology on $\mathcal{C}$. If the class $we(\mathcal{T})$ of $\mathcal{T}$-equivalences has the 2-out-of-3 property and $\mathcal{C}$ has enough $\mathcal{E}_{\mathcal{T}}$-projectives, then*

   $$(Cat(\mathcal{C}), fib(\mathcal{T}), cof(\mathcal{T}), we(\mathcal{T}))$$

   is a model category.

2. *An internal category (A_0, A_1) is cofibrant if and only if A_0 is $\mathcal{E}_{\mathcal{T}}$-projective.*

We apply this to $\mathcal{C} = \mathbf{Cat}$ so that $Cat(\mathcal{C}) = \mathbf{DblCat}$.

Results on $Cat(\mathbf{Cat}) = \mathbf{DblCat}$

Theorem 12 *(FPP) Let τ be the Grothendieck topology where a basic cover of $B \in \mathcal{C}$ is*

$$\{F : A \to B\}$$

such that $(NF)_k$ is surjective for all $k \geq 0$. Then τ induces a model structure on $\mathbf{DblCat}$.

Theorem 13 *(FPP) The model structure induced by τ is the same as the transferred levelwise categorical structure from $[\Delta^{op}, \mathbf{Cat}]$.*

Results on $Cat(\mathbf{Cat}) = \mathbf{DblCat}$

Theorem 14 *(FPP) Let τ' be the Grothendieck topology where a basic cover of $B \in \mathcal{C}$ is*

$$\{F : A \to B\}$$

such that F is surjective on objects and full. Then τ' induces a model structure on $\mathbf{DblCat}$.

Corollary 15 *(FPP) In this model structure, a double category $\mathbb{D}$ is cofibrant if and only if $\mathbb{D}_0$ is projective with respect to functors that are surjective on objects and full.*

Remark 16 *Embed* **2-Cat** *vertically in* **DblCat**. *Then a 2-category is cofibrant in Lack's model structure if and only if it is cofibrant in the τ' structure.*

2-Monads and **2-Cat**

The adjunction

$$Cat(\mathbf{Graph}) \quad \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \quad Cat(\mathbf{Cat})$$

is monadic and induces a 2-monad on $Cat(\mathbf{Graph})$ whose algebras are double categories.

Proposition 17 *(FPP) The model structure on* **DblCat** *induced by this 2-monad as prescribed by Lack is the τ' model structure.*

Proposition 18 *(FPP) Embed* **2-Cat** *vertically in* **DblCat**. *If a 2-functor is a cofibration in* **DblCat**, *then it is a cofibration in* **2-Cat**.

Summary of Main Results

We have transferred the two levelwise model structures on $[\Delta^{op}, \mathbf{Cat}]$ via

$$[\Delta^{op}, \mathbf{Cat}] \underset{N_h}{\overset{c_h}{\rightleftarrows}} \mathbf{DblCat}\,. \quad (\perp)$$

We have shown that the Reedy categorical structure does not transfer.

We also constructed the transferred categorical model structure using the methods of Everaert, Kieboom, and Van der Linden, and obtained another structure from categorically surjective functors.