# Higher Hopf formulae

$\forall n \geq 1$

$$H_{n+1}(A, \mathcal{B}) = \frac{[P_{\mathrm{init}}]_0 \cap \bigcap_{i=0}^{n-1} K[p_i]}{[P]_n}$$

- $\mathcal{A}$ is a semi-abelian category
- $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$
- $A \in |\mathcal{A}|$
- $P$ is any $n$-fold presentation of $A$, i.e. $P \in |\mathsf{Ext}^n \mathcal{A}|$ with $P_{\mathrm{termin}} = A$ and all objects projective, except $A$
- The $p_i$ are the "initial ribs" of the $n$-dimensional cube $P$

**Theorem**
- $\mathcal{A}$ semi-abelian, monadic over Set
- $I : \mathcal{A} \to \mathcal{B}$ reflector for Birkhoff subcategory
- $\mathbb{G}$ comonad induced by $\mathcal{A} \rightleftarrows \text{Set}$
- $A \in |\mathcal{A}|$

$$H_{n+1}(A, I)_{\mathbb{G}} \cong H_{n+1}(A, \mathcal{B})_{\text{Hopf}}$$

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$$0 \to K \rightarrowtail B \xrightarrow{f} A \to 0$$

$A \in |\text{Ext}^{n-1}\mathcal{A}|$
$f \in |\text{Ext}^{n}\mathcal{A}|$
$B$ projective

**Proposition** "Base step" $n \geq 1$

$$H_2(A, I_{n-1})_{\mathbb{G}_{n-1}} \cong \frac{[B,B]_{n-1} \cap K}{[B,K]_{n-1}} = H_2(A, \mathcal{B}_{n-1})_{\text{Hopf}}$$

**Proposition** "Induction step" $n \geq 1$ $k \geq 2$

$$H_k(f, I_n)_{\mathbb{G}_n} \cong \left( H_{k+1}(A, I_{n-1})_{\mathbb{G}_{n-1}} \to 0 \right)$$
$$= \iota H_{k+1}(A, I_{n-1})_{\mathbb{G}_{n-1}}$$

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$\iota^n : \mathcal{A} \to \text{Arr}^n\mathcal{A}$ $\qquad \iota^0 A = A$

$A \mapsto (\iota^{n-1}A \to 0)$

## Some examples

$f$ $n$-fold presentation of $A$

$$H_{n+1}(A, I)_{\mathcal{G}} \cong \frac{[f_{init}, f_{init}] \cap \bigcap_{i=0}^{n-1} K[f_i]}{\prod_{S \subseteq n} \left[ \bigcap_{i\in S} K[f_i], \bigcap_{i\notin S} K[f_i] \right]}$$

$I : Gp \to Ab$ $\quad [-,-]$ commutator

$I : Lie \to AbLie$ $\quad [-,-]$ Lie bracket

$I : CRng \to ZeroRng$ $\quad [-,-] = - \cdot -$ product

$I : PXMod \to XMod$ $\quad [-,-] = \langle -,- \rangle$ Peiffer commutator

$I : Gp \to Nil_k$ $\quad \overline{[-,-]}$ commutator

$$H_{n+1}(A, I)_{\mathcal{G}} \cong \frac{[\cdots[f_{init}, f_{init}], f_{init}], \cdots] \cap \bigcap_{i=0}^{n-1} K[f_i]}{\prod_{I_1 \cup \cdots \cup I_k = n} \left[\cdots\left[\left[\bigcap_{i\in I_1} K[f_i], \bigcap_{i\in I_2} K[f_i]\right], \bigcap_{i\in I_3} K[f_i]\right], \cdots\right]}$$