Equality of mathematical structures

Martín Hötzel Escardó
School of Computer Science
University of Birmingham, UK

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Independently of any choice of foundation, we regard two groups to be the same, for all mathematical purposes, if they are isomorphic. Likewise, we consider two topological spaces to be the same if they are homeomorphic, two metric spaces to be the same if they are isometric, two categories to be the same if they are equivalent, and so on.

Do we *choose* these notions of sameness, motivated by particular mathematical applications, or are these notions of sameness imposed upon us, independently of what we want to do with groups, topological spaces, metric spaces and categories? This may be regarded as a philosophical question. However, if

- we adopt Martin-Löf type theory as our mathematical foundation,
- take the notion of equality to be the identity type, and
- assume Voevodsky's univalence axiom,

then

- we can *prove* that equality of groups *is* isomorphism, equality of topological spaces *is* homeomorphism, equality of metric spaces *is* isometry, equality of categories *is* equivalence etc.

For a large class of algebraic structures, this was first proved by Coquand and Danielsson [2]. For objects of categories, this was formulated by Aczel [4], and for categories themselves this was proved by Ahrens, Kapulkin and Shulman [1].

This talk will be mainly expository, based on our lecture notes [3]. A minor novelty will be a general theorem and various specializations that can be used to attack all the above characterizations of equality, and more, in a uniform and modular way.

References

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