Fraction, Restriction and Range Categories From Non-Monic Classes of Morphisms

Walter Tholen

Department of Mathematics and Statistics, York University, Toronto, Canada
tholen@mathstat.yorku.ca

For a composition-closed and pullback-stable class $\mathcal{S}$ of morphisms in a category $\mathcal{C}$ containing all isomorphisms, we form the category $\mathsf{Span}(\mathcal{C},\mathcal{S})$ of $\mathcal{S}$-spans (s, f) in $\mathcal{C}$ with first "leg" s lying in $\mathcal{S}$, and give an alternative construction of its quotient category $\mathcal{C}[\mathcal{S}^{-1}]$ of $\mathcal{S}$-fractions. Instead of trying to turn $\mathcal{S}$-morphisms "directly" into isomorphisms, we turn them separately into retractions and into sections in a universal manner, thus obtaining the quotient categories $\mathsf{Retr}(\mathcal{C},\mathcal{S})$ and $\mathsf{Sect}(\mathcal{C},\mathcal{S})$. The fraction category $\mathcal{C}[\mathcal{S}^{-1}]$ is their largest joint quotient category.

Without confining $\mathcal{S}$ to be a class of monomorphisms of $\mathcal{C}$, we show that $\mathsf{Sect}(\mathcal{C},\mathcal{S})$ admits a quotient category, $\mathsf{Par}(\mathcal{C},\mathcal{S})$, whose name is justified by two facts. On one hand, for $\mathcal{S}$ a class of monomorphisms in $\mathcal{C}$, it returns the category of $\mathcal{S}$-spans in $\mathcal{C}$, also called $\mathcal{S}$-partial maps in this case; on the other hand, under a mild additional hypothesis on $\mathcal{S}$, but without constraining $\mathcal{S}$ to be a class of monomorphisms in $\mathcal{C}$, one can show that $\mathsf{Par}(\mathcal{C},\mathcal{S})$ is a split restriction category (in the sense of Cockett and Lack). A further quotient construction produces even a range category (in the sense of Cockett, Guo and Hofstra), $\mathsf{RaPar}(\mathcal{C},\mathcal{S})$, which is still large enough to admit $\mathcal{C}[\mathcal{S}^{-1}]$ as its quotient.

Both, Par and RaPar, are the left adjoints of global 2-adjunctions. When restricting these to their "fixed objects", one obtains precisely the 2-equivalences by which their name givers characterized restriction and range categories, as categories $\mathcal{C}$ suitably structured by a class $\mathcal{S}$ of *mono*morphisms in $\mathcal{C}$. Hence, their mono constraint for the classes $\mathcal{S}$ emerges naturally in our more general context.

References

[1] F. Borceux: *Handbook of Categorical Algebra 1, Basic Category Theory.* Cambridge University Press, Cambridge 1994.

[2] J.R.B. Cockett, X. Guo, P. Hofstra: Range Categories I: General Theory. *Theory and Applications of Categories* 26 (17), 412–452, 2012.

[3] J.R.B. Cockett and S. Lack: Restriction Categories I. *Theoretical Computer Science* 270, 223–259, 2002.

[4] P. Gabriel, M. Zisman: *Calculus of Fractions and Homotopy Theory.* Springer-Verlag, Berlin-Heidelberg-NewYork 1967.

[5] S.N. Hosseini, A.R. Shir Ali Nasab, W. Tholen: Abandoning monomorphisms: partial maps, fractions and factorizations. arXiv:1903.00081 [math.CT].

[6] H. Schubert: *Categories.* Springer-Verlag, Berlin-Heidelberg-New York 1972.

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