The third cohomology group classifies double central extensions

Tim Van der Linden
Joint work with Diana Rodelo

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Connecting two branches of non-abelian (co)homology

Bourn–Rodelo direction approach to cohomology

Bourn, *Baer sums and fibered aspects of Mal'cev operations* (1999)

Bourn, *Aspherical abelian groupoids and their directions* (2002)

Bourn, *Baer sums in homological categories* (2007)

Bourn–Rodelo, *Cohomology without projectives* (2007)

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Semi-abelian homology via categorical Galois theory

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The second cohomology group classifies central extensions

$H^2(Z, A) \cong \mathsf{Centr}^1(Z, A)$ for group Z and abelian group A

A central extension f of Z by A induces a short exact sequence

$$0 \longrightarrow A \overset{\ker f}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Z \longrightarrow 0$$

where $axa^{-1}x^{-1} = 1$ for all $a \in A$ and $x \in X$.

- $\mathsf{Centr}^1(Z, A) = \{$equivalence classes of central extensions$\}$
- group structure on $\mathsf{Centr}^1(Z, A)$: Baer sum

Generalised to semi-abelian context

Bourn–Janelidze, *Extensions with abelian kernels in protomodular categories* (2004)

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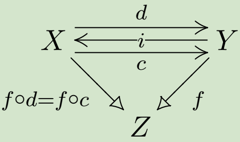
The third cohomology group $H^3(Z, A)$

Definition (Bourn, Bourn–Rodelo)

Let $\mathcal{A}$ be a Moore category, Z an object of $\mathcal{A}$. An **aspherical internal groupoid** in $\mathcal{A} \downarrow Z$ is a groupoid



(A)

in $\mathcal{A} \downarrow Z$ with f and $(d, c) \colon X \to Y \times_Z Y$ regular epic.

- category $\mathsf{Asph}(\mathcal{A} \downarrow Z)$

The third cohomology group $H^3(Z, A)$

Definition (Bourn, Bourn–Rodelo)

- The **direction functor** $\partial\colon \mathsf{Asph}(\mathcal{A} \downarrow Z) \to \mathsf{Mod}_Z\mathcal{A}$ maps $\mathbf{A}$ to the Z-module (A, ξ) corresponding to (p, s) in

$$\begin{array}{ccc} R[(d,c)] & \longrightarrow & Z \ltimes (A,\xi) \\ {\scriptstyle \mathrm{pr}_0}\downarrow\uparrow{\scriptstyle (1_X,1_X)} & & {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} \\ X & \xrightarrow[f \circ d]{} & Z. \end{array}$$

- The **third cohomology group of Z with coefficients in** (A, ξ) is the group $H^3(Z, (A, \xi)) = \pi_0\partial^{-1}(A, \xi)$.
- If ξ is the trivial action then (A, ξ) is just the abelian object A and we denote $H^3(Z, (A, \xi)) = H^3(Z, A)$.
- Note that $A = K[p] = K[\mathrm{pr}_0] = K[(d, c)] = K[d] \cap K[c]$.

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Double central extensions: Γ-covers

Central extensions, in general

$\mathcal{A}$ semi-abelian, $\mathcal{B}$ Birkhoff subcategory give Galois structure

$$\Gamma_0 = (\mathcal{A} \underset{\supset}{\overset{I}{\rightleftarrows}} \mathcal{B}, |\mathsf{Ext}\mathcal{A}|, |\mathsf{Ext}\mathcal{B}|)$$

- $\mathsf{Ext}\mathcal{A} = (\{\text{regular epis in } \mathcal{A}\}, \{\text{commutative squares}\})$
- central extensions: $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$

Double central extensions, in general

Galois structure

$$\Gamma = (\mathsf{Ext}\mathcal{A} \underset{\supset}{\overset{I_1}{\rightleftarrows}} \mathsf{CExt}_{\mathcal{B}}\mathcal{A}, |\mathsf{Ext}^2\mathcal{A}|, |\mathsf{Ext}^2\mathcal{B}|)$$

- notion of double central extension = Γ-cover
- used in computation of $H_3(-, I)$

Double central extensions: Γ-covers

Central extensions, w.r.t. abelianisation

$\mathcal{A}$ semi-abelian, $\mathcal{B} = \mathsf{Ab}\mathcal{A}$ give Galois structure

$$\Gamma_0 = (\mathcal{A} \underset{\supset}{\overset{\mathrm{ab}}{\rightleftarrows}} \mathsf{Ab}\mathcal{A}, |\mathsf{Ext}\mathcal{A}|, |\mathsf{Ext}\mathsf{Ab}\mathcal{A}|)$$

- $\mathsf{Ext}\mathcal{A} = (\{\text{regular epis in } \mathcal{A}\}, \{\text{commutative squares}\})$
- central extensions: $|\mathsf{CExt}\mathcal{A}| = \{f \in |\mathsf{Ext}\mathcal{A}| \mid [R[f], \nabla] = \Delta\}$

Double central extensions, w.r.t. abelianisation

Galois structure

$$\Gamma = (\mathsf{Ext}\mathcal{A} \underset{\supset}{\overset{\mathrm{centr}}{\rightleftarrows}} \mathsf{CExt}\mathcal{A}, |\mathsf{Ext}^2\mathcal{A}|, |\mathsf{Ext}^2\mathsf{Ab}\mathcal{A}|)$$

- notion of double central extension = Γ-cover
- used in computation of $H_3(-, \mathrm{ab})$

Double extensions: aspherical spans in $\mathcal{A} \downarrow Z$

Definition

A span $(X, d, c) = \begin{array}{ccc} & X & \\ {}^{d}\swarrow & & \searrow^{c} \\ D & & C \end{array}$ in a regular category $\mathcal{A}$

- **has global support** when $!_D \colon D \to 1$ and $!_C$ regular epic;
- **is aspherical** when also $(d, c) \colon X \to D \times C$ is regular epic.

Proposition

A commutative square in $\mathcal{A}$ is a double extension if and only if it represents an aspherical span in $\mathcal{A} \downarrow Z$.

$$\begin{array}{ccc} X & \xrightarrow{\;c\;} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[\;f\;]{} & Z \end{array}$$

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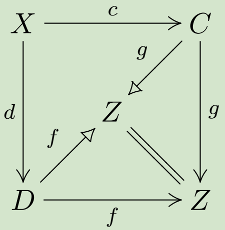

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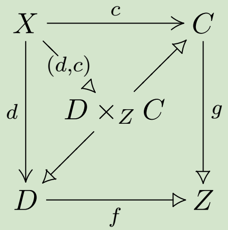

Double central extensions: commutator version

Theorem (Janelidze, Gran–Rossi)

In a Mal'tsev variety $\mathcal{A}$, a double extension

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

is central if and only if

1. $[R[d], R[c]] = \Delta_X$;
2. $[R[d] \cap R[c], \nabla_X] = \Delta_X$.

Double central extensions: internal pregroupoids, $[R[d], R[c]] = \Delta_X$

Definition (Kock, Johnstone, Pedicchio)

Let $\mathcal{A}$ be a finitely complete category. A **pregroupoid** or **herdoid** (X, d, c, p) in $\mathcal{A}$ is a span (X, d, c)

$$\begin{array}{ccc} & X & \\ {}^{d}\swarrow & & \searrow^{c} \\ D & & C \end{array}$$

with a partial ternary operation p on X satisfying

1. $p(\alpha, \beta, \gamma)$ is defined iff $c(\alpha) = c(\beta)$ and $d(\gamma) = d(\beta)$;
2. $dp(\alpha, \beta, \gamma) = d(\alpha)$ and $cp(\alpha, \beta, \gamma) = c(\gamma)$;
3. $p(\alpha, \alpha, \gamma) = \gamma$ and $p(\alpha, \gamma, \gamma) = \gamma$;
4. $p(\alpha, \beta, p(\gamma, \delta, \epsilon)) = p(p(\alpha, \beta, \gamma), \delta, \epsilon)$.

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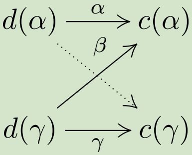

$$p(\alpha, \beta, \gamma) \colon d(\alpha) \to c(\gamma)$$

Double central extensions: internal pregroupoids, $[R[d], R[c]] = \Delta_X$

Proposition

- *A pregroupoid structure* $p\colon R[d] \times_X R[c] \to X$ *on a span* (X, d, c) *is a cooperator between* $R[d]$ *and* $R[c]$.
- *In a semi-abelian category, a span* (X, d, c) *carries at most one pregroupoid structure—precisely when* $[R[d], R[c]] = \Delta_X$.
- *A double extension*

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

satisfies $[R[d], R[c]] = \Delta_X$ *iff the aspherical span* (X, d, c) *in* $\mathcal{A} \downarrow Z$ *is an internal pregroupoid.*

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Double central extensions: $[R[d] \cap R[c], \nabla_X] = \Delta_X$

Definition

An aspherical pregroupoid (X, d, c) in $\mathcal{A} \downarrow Z$ is **central** when $(d, c) \colon X \to D \times_Z C$ is a central extension in $\mathcal{A}$.

Proposition

Let

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

be a commutative square in a semi-abelian category $\mathcal{A}$ such that (X, d, c) is an aspherical pregroupoid in $\mathcal{A} \downarrow Z$. Then this square is a double central extension if and only if (X, d, c) is a central pregroupoid in $\mathcal{A} \downarrow Z$.

Double central extensions: equivalent conditions

Theorem

Let, in a semi-abelian category $\mathcal{A}$,

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array} \qquad (\mathbf{B})$$

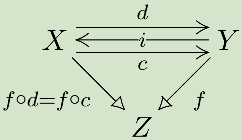
be a double extension. The following are equivalent:

1. $\mathbf{B}$ *is a double central extension;*
2. $[R[d], R[c]] = \Delta_X$ *and* $[R[d] \cap R[c], \nabla_X] = \Delta_X$*;*
3. (X, d, c) *is a central pregroupoid in* $\mathcal{A} \downarrow Z$*.*

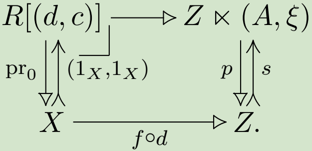
The third cohomology group $H^3(Z, A)$

Definition (Bourn, Bourn–Rodelo)

The **direction functor** $\partial\colon \mathsf{Asph}(\mathcal{A} \downarrow Z) \to \mathsf{Mod}_Z\mathcal{A}$ maps



to the Z-module (A, ξ) corresponding to (p, s) in



Double central extensions: the direction functor

Proposition

An aspherical groupoid (X, d, c) in $\mathcal{A} \downarrow Z$ has for direction a trivial module $A = (A, \xi)$ if and only if (X, d, c) is central.

Proof

$$\begin{array}{ccc} R[(d,c)] & \longrightarrow & Z \ltimes (A, \xi) \\ \mathrm{pr}_0 \downarrow \uparrow (1_X, 1_X) & & p \downarrow \uparrow s \\ X & \xrightarrow[f \circ d]{} & Z. \end{array}$$

$(p, s) = (\mathrm{pr}_Z, (1_Z, 0)) \colon Z \times A \leftrightarrows Z$

$\Leftrightarrow \quad (\mathrm{pr}_0, (1_X, 1_X)) = (\mathrm{pr}_X, (1_X, 0)) \colon X \times A \leftrightarrows X$

$\Leftrightarrow \quad R[(d,c)] = R[d] \cap R[c]$ is central

$\Leftrightarrow \quad [R[d] \cap R[c], \nabla_X] = \Delta_X$

Double central extensions: the direction functor

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\end{array}
$$

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Definition

- The **direction functor** $\partial\colon \mathsf{CExt}^2_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}$ maps a double central extension

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

  of Z to $A = K[d] \cap K[c]$.

- $\mathsf{Centr}^2(Z, A) = \pi_0 \partial^{-1} A$ is the set of equivalence classes of double central extensions **of** Z **by** A.

Double central extensions: the direction functor

Double central extensions of Z by A are 3×3 diagrams

$$
\begin{array}{ccccc}
A & \rightarrowtail & K[d] & \twoheadrightarrow & K[g] \\
\downarrow & & \downarrow & & \downarrow \\
K[c] & \rightarrowtail & X & \xrightarrow{c} & C \\
\downarrow & & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\
K[f] & \rightarrowtail & D & \xrightarrow[f]{} & Z
\end{array}
$$

- A is the direction of $\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$

Double central extensions: the group $\mathsf{Centr}^2(Z, A)$

Definition

$\mathsf{Centr}^2(Z, A) = \pi_0 \partial^{-1} A$ is the set of equivalence classes of double central extensions of Z by A.

Proposition

We have a finite product-preserving functor

$$\mathsf{Centr}^2(Z, -) \colon \mathsf{Ab}\mathcal{A} \to \mathsf{Set},$$

hence a functor

$$\mathsf{Centr}^2(Z, -) \colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}.$$

Double central extensions: a representing groupoid

Definition of a bijection $\mathsf{Centr}^2(Z, A) \to H^3(Z, A)$

We map a central pregroupoid (X, d, c) in $\mathcal{A} \downarrow Z$ to its associated central groupoid in $\mathcal{A} \downarrow Z$: via the pullback

$$\begin{array}{ccc} Y & \xrightarrow{\;p\;} & X \\ {\scriptstyle (\mathrm{dom},\mathrm{cod})}\downarrow & \lrcorner & \downarrow{\scriptstyle (d,c)} \\ X \times_Z X & \xrightarrow[d \times_Z c]{} & D \times_Z C. \end{array}$$

- $(Y, \mathrm{dom}, \mathrm{cod})$ is a groupoid, $(\mathrm{dom}, \mathrm{cod})$ is a central extension in $\mathcal{A}$ and $\partial(p, d, c) = 1_A$

Conclusion

Theorem

The third cohomology group classifies double central extensions:

If Z is an object and A is an abelian object in a Moore category, then

$$H^3(Z, A) \cong \mathsf{Centr}^2(Z, A).$$

Conjecture

The $(n+1)$-st cohomology group classifies n-fold central extensions:

If Z is an object and A is an abelian object in a Moore category, then

$$H^{n+1}(Z, A) \cong \mathsf{Centr}^n(Z, A)$$

for all $n \geq 1$.

Conclusion

Theorem

The third cohomology group classifies double central extensions:

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