Categorical logics extending Birkhoff's equational logic

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- J. Adámek, M. Sobral and L. Sousa, *A Logic of Implications in Algebra and Coalgebra*, Algebra Universalis (2008)
- J. Adámek, M. Hébert and L. Sousa, *A Logic of Orthogonality*, Archivum Mathematicum (2006)
- J. Adámek, M. Hébert and L. Sousa, *A Logic of Injectivity*, J. of Homot. and Rel. Struct. (2007)
- J. Adámek, M. Hébert and L. Sousa, *The Orthogonal Subcategory Problem and the Small Object Argument*, Applied Cat. Struct. (2008)

Birkhofft's deduction system for equations

(1) $$\frac{}{u = u}$$

(2) $$\frac{u = v}{v = u}$$

(3) $$\frac{u = v,\ v = w}{u = w}$$

(4) $$\frac{u_1 = v_1,\ u_2 = v_2,\ \ldots,\ u_n = v_n}{f(u_1, u_2, \ldots, u_n) = f(v_1, v_2, \ldots, v_n)},$$

for all n-ary symbols f in Σ.

(5) $$\frac{u = v}{u^\sigma = v^\sigma},$$ *for all substitutions σ.*

Deduction system for implications

(R. Quackenbush, *Proc. Amer. Math. Society* 103 (1988))

Axiom 1: $$\frac{}{\mathcal{P} \Rightarrow u = v}$$ *if* $\mathcal{P}$ *contains* $u = v$

Axiom 2: $$\frac{}{\mathcal{P} \Rightarrow u = u}$$

Symmetry: $$\frac{\mathcal{P} \Rightarrow u = v}{\mathcal{P} \Rightarrow v = u}$$

Transitivity: $$\frac{\mathcal{P} \Rightarrow u = v,\ \mathcal{P} \Rightarrow v = w}{\mathcal{P} \Rightarrow u = w}$$

Congruence: $$\frac{\mathcal{P} \Rightarrow u_1 = v_1,\ \ldots,\ \mathcal{P} \Rightarrow u_n = v_n}{\mathcal{P} \Rightarrow f(u_1, \ldots, u_n) = f(v_1, \ldots, v_n)}$$

Invariance: $$\frac{\mathcal{P} \Rightarrow u = v}{\mathcal{P}^\sigma \Rightarrow u^\sigma = v^\sigma}$$

Cut: $$\frac{\mathcal{P} \Rightarrow s_i = t_i (i = 1, \ldots, n),\ \{s_i = t_i\}_{i=1}^n \Rightarrow u = v}{\mathcal{P} \Rightarrow u = v}$$

Finitary Injectivity Logic

Formulas: finitary epimorphisms

Semantic: A is a model of f if A is injective w.r.t. f

Inference rules:

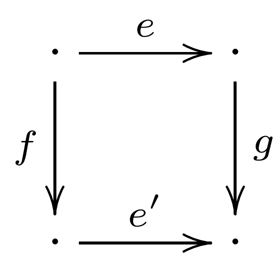
IDENTITY
$$\frac{}{\mathrm{id}_A}$$

COMPOSITION
$$\frac{e,\ e'}{e' \cdot e}$$

CANCELLATION
$$\frac{e' \cdot e}{e}$$

PUSHOUT
$$\frac{e}{e'}$$

for every pushout



Finitary Orthogonality Logic

Formulas: finitary morphisms

Semantic: A is a model of f if A is orthogonal to f

Inference rules:

IDENTITY

COMPOSITION

PUSHOUT

WEAK CANCELLATION

$$\frac{u \cdot t \quad v \cdot u}{t}$$

COEQUALIZER

$$\frac{s}{t}$$

if $\underset{g}{\overset{f}{\rightrightarrows}} \xrightarrow{t}$ is a coeq. s. t. $f \cdot s = g \cdot s$

Injectivity Logic

Formulas: morphisms

Semantic: A is a model of f if A is injective w.r.t. f

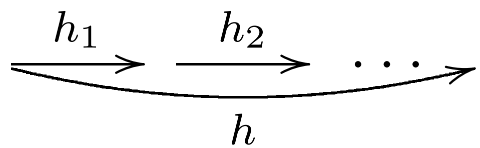
Inference rules:

IDENTITY

$$\frac{}{\mathrm{id}_A}$$

TRANSFINITE COMPOSITION

$$\frac{h_i,\ i \in \alpha}{h}$$



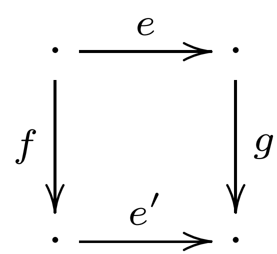
CANCELLATION

$$\frac{e' \cdot e}{e}$$

PUSHOUT

$$\frac{e}{e'}$$

for every pushout



In locally presentable categories,
the Injectivity Logic and the Orthogonality Logic are always complete for sets of morphisms and, moreover, for quasi-presentable classes.

For classes in general:

The completeness of the Orthogonality Logic for all classes of morphisms is equivalent to the Vopěnka Principle.

The Injectivity Logic is not always complete (independently of set-theorical large cardinals principles).