EXTERNAL DERIVATIONS OF INTERNAL GROUPOIDS

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Let G be a group and $\varphi\colon G \to AutH$ a G-group. A derivation of G in H is a map $d\colon G \to H$ such that $d(xy) = d(x) + x \cdot d(y)$. If H is a G-module, i.e. if H is abelian, the set $Der(G,H)$ of derivations is an abelian group w.r.t. the point-wise sum. If H is not abelian, in general $Der(G,H)$ is just a pointed set. J.H.C. Whitehead discovered the following fact.

0.1. Theorem. *Let* $(H \xrightarrow{\partial} G \xrightarrow{\varphi} AutH)$ *be a crossed module of groups. The set* $Der(G,H)$ *is a monoid w.r.t.* $(d_1 + d_2)(x) = d_1(\partial(d_2(x))x) + d_2(x)$.

The aim of this talk is to understand in a more conceptual way Whitehead product of derivations. The idea is to replace crossed modules of groups by the equivalent notion of internal groupoids in the category of groups. Using the language of internal groupoids, Whitehead product becomes clear: it is nothing but the composition in the internal category. The surprise is that, once expressed in terms of internal groupoids, Whitehead theorem, as well as some other basic properties of derivations, has nothing to do with groups, but holds in the very general context of internal groupoids in an arbitrary category $\mathcal{G}$ with finite limits. In this way, these results hold not only for crossed module of groups (when $\mathcal{G}$ is the category of groups), but also for crossed modules of Lie algebras (take for $\mathcal{G}$ the category of Lie algebras), Lie groupoids (take for $\mathcal{G}$ the category of smooth manifolds), and of course ordinary groupoids (take for $\mathcal{G}$ the category of sets).

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