Duality-TV

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Over the past few years the work of several authors was concerned with the study of topological structures as generalised categories. This research program has its roots in Lawvere's [10] observation that both ordered sets and metric spaces can be viewed as quantale-enriched categories, and in Barr's [1] description of topological spaces as relational algebras for the ultrafilter monad. The latter result allows us to interpret a topological space as a generalised ordered set (or better: generalised enriched category) via its ultrafilter convergence relation $UX \times X \to 2$. Both approaches are brought under one roof with the notion of $(\mathbb{T}, \mathsf{V})$-category, for a Set-monad $\mathbb{T}$ and a commutative quantale V (see, for instance, [2, 7]). Topological spaces appear as $(\mathbb{U}, 2)$-categories and metric spaces as $(\mathbb{I}, [0, \infty])$-categories where $\mathbb{I}$ denotes the identity monad, and further examples of $(\mathbb{T}, \mathsf{V})$-categories include approach spaces (as $(\mathbb{U}, [0, \infty])$-categories), probabilistic metric spaces and multi-ordered sets. This categorical setting for topological objects permits now to transfer ideas in both directions: we study spaces using tools like module, weighted (co)limit, the Yoneda Lemma, Kan extension and adjoint functor (see [3, 8, 4]); and for instance closure operators [5] turn out to be very helpful for investigating (quantale-enriched) categories. In this talk we recall some of these ideas and use them in the study of duality theory for (topological and approach) spaces.

In earlier talks we made already the observation that the well-known adjunction

$$\begin{tikzcd} \mathsf{Ord} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "G"] \end{tikzcd}, \quad F(X) = 2^X \text{ or } F(X) = 2^{X^{\mathrm{op}}}$$

between the category Ord of ordered sets and monotone maps and the dual of the category CCD of constructive completely distributive lattices and left and right adjoint monotone maps leads to two rather different constructions for $(\mathbb{T}, \mathsf{V})$-categories. For instance, $X \mapsto \mathsf{Top}(X, 2)$ gives the "usual" dual adjunction between Top and the category Frm of frames and frame homomorphisms, but $X \mapsto \mathsf{Top}(UX, 2)$ results in a dual adjunction between Top and the category CDTop of "completely distributive" topological spaces and left and right adjoint continuous maps. Here one can topologise UX in such a way that the resulting space is exponentiable and behaves like the "dual category", and therefore we can think of a continuous map $\varphi : UX \to 2$ as a "down-set" $\varphi : X^{\mathrm{op}} \to 2$ of X. Moreover, $X \mapsto 2^{X^{\mathrm{op}}}$ gives rise to the "down-set" monad on Top which turns out to be isomorphic to the filter monad. Although different in nature, both adjunctions are actually the same as CDTop can be shown to be equivalent to Frm. Nevertheless, the second construction seems to be closer to the Ord-case and allows us to go further.

In [11] R. Rosebrugh and R. J. Wood showed that the category $\mathsf{CCD}_{\mathrm{sup}}$ of constructive complete distributive lattices and suprema preserving maps is equivalent to the Karoubian envelope of the category Rel of sets and relations. This theorem turned out to be very powerful since it synthesises many facts about complete distributive lattices, implies various known duality theorems in lattice theory (for example, $\mathsf{Set}^{\mathrm{op}} \cong \mathsf{CABool}$ and $\mathsf{Ord}^{\mathrm{op}} \cong \mathsf{TAL}$), and allows to transfer nice properties and structures from Rel to $\mathsf{CCD}_{\mathrm{sup}}$. Later on, in [12] they observed that this theorem is not really about lattices but rather a special case of a much more general result about "a mere monad D on a mere category C". The equivalence above appears now for both the power-set monad on Set and the down-set monad on Ord. Most importantly for us, [12] can be applied to the "down-set monad" on $(\mathbb{T}, \mathsf{V})$-categories and therefore paves the road towards similar results for topological, metric and approach spaces. For instance, the category $\mathsf{CDTop}_{\mathrm{sup}}$ of "completely distributive" topological spaces and left adjoint continuous maps is dually equivalent to the Karoubian envelope of a suitable category of convergence relations, and a corresponding result holds for approach spaces. We explain how this entails the well-known duality between spatial frames and sober spaces, but also how it describes the dual of Frm as a category of "non-reflexive" sober spaces. Finally, many interesting classes of topological spaces (stably compact spaces, ordered compact Hausdorff spaces,...) arise as the algebras for a submonad of the filter monad on Top, which in the language of categories corresponds to a choice of a class of "down-sets" (see [4]). Each such choice leads to a series of duality theorems for $(\mathbb{T}, \mathsf{V})$-categories in general and therefore for topological and approach spaces in particular.

If time permits, we will also discuss the Lawson duality [9] for quantale enriched categories.

References

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