Lattice-valued categories

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The notion of category comprises and reflects the fundamental concepts of partially (pre-)ordered classes and monoidal structures. Going beyond the purely technical aspects, the concept of category merges two basic connectives of logic; in fact the notion of morphism is related to the implication connective of deductive systems and their composition depends on the adjoint connective of conjunction (the residuum of implication). So it is not surprising that the notion of category has met for long time the basic algebraic structures of logics, either classical or not.

This has led on one side to an enrichment of the notion of category; in particular, the notion of (monoidal) closed category was introduced by Eilenberg and Kelly [1] in 1966 and it was widely promoted by Lowvere [4], since the early seventies, as a fundamental tool of a categorical approach to logics. These kinds of categories generalize in some sense and play a similar role as the implicative, possibly residuated, structures and are the base over which enriched categories are built.

On another side, the notion of category has been extended from the viewpoint of lattice-valued mathematics with a process of so-called fuzzyfication bringing to a notion of lattice-valued category. The original approach of Goguen [2] (1967) has been further developed by Sostak [5] et al. in the last decade.

In any case, the classical concepts of categories and functors are assumed as basic concepts and as a starting point of their generalizations, enrichments and extensions.

The present approach does not assume the notion of category as modeling the classical set theory but tries to merge in a lattice-valued structure the fundamental concepts of lattice-valued class and of lattice-valued order. The algebraic feature of the hom-sets with respect to the composition of morphisms depends on the structure of the base-lattice considered. We use as a base-lattice a kind of complete implicative algebra L with an adjoint product, introduced in [3].

An L-category has an L-class of objects, an L-class of morphisms and suitable L-relations modeling the notions of domain, codomain and composition of morphisms. Identity morphisms and associativity of the composition are also expressed by means of suitable L-relations. It is seen that the L-sets, the L-ordered L-sets and the implicative algebra L itself provide examples of L-categories.

Also L-functors between L-categories are defined as pairs of L-relations well behaved with the structure of the L-categories. Examples are provided, also in the case when the L-categories are in fact those determined either by L, or by an L-ordered set or by an L-ordered L-set; in these cases, the assumption of crispness forces the

L-functors to be pairs of functions that can be viewed as morphisms of implicative algebras, or of L-ordered L-sets or of L-ordered L-sets.

This approach to categories and functors in the context of lattice-valued mathematics allows a critical view to some aspects of the way these concepts have been classically expressed.

References

[1] S. Eilenberg, G.M. Kelly: *Closed categories*, in: S. Eilenberg, D.K. Harrison (Eds), Proceedings of the Conference on Categorical Algebra (la Jolla, 1965), Springer, Berlin, 1966, pp. 421–562.

[2] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. 18 (1967), 145–174.

[3] C. Guido, P. Toto: *Extended-order algebras*, Journal of Applied Logic, 6 (2008), 609–626.

[4] R. Lowvere: *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano 43 (1973), 135–166.

[5] A. P. Sostak, *L-valued categories: generalities and examples related to algebra and topology*, in W. Gähler, G. Preuss (Eds): Categorical Structures and their Applications, World Scientific Publishing, 2004, 291–311.