Closure operators and radicals

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Using Dikranjan-Giuli notion of closure operator w.r.t. $\mathcal{M}$ [3] and a well-behaved class of $\mathcal{M}$-subobjects, we present a generalization of preradical which allows us to study simultaneously algebraic torsion theories (see [2, 1]) and factorization systems.

Namely, we show that the following two results are particular instances of a more general result.

Theorem 1. *For a pair $(\mathcal{A}, \mathcal{B})$ of subclasses of $\mathcal{M}$ the following assertions are equivalent:*

(i) *$(\mathcal{A}, \mathcal{B})$ is a factorization system for morphisms of $\mathcal{M}$;*

(ii) *there is a weakly hereditary idempotent closure operator c such that $\mathcal{A}$ is the class of c-dense subobjects and $\mathcal{B}$ is the class of c-closed subobjects.*

Theorem 2. *For a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of an abelian category the following assertions are equivalent:*

(i) *$(\mathcal{T}, \mathcal{F})$ is a torsion theory;*

(ii) *there is an idempotent radical r such that $\mathcal{T}$ is the class of r-torsion objects and $\mathcal{F}$ is the class of r-torsion-free objects.*

References

[1] D. Bourn and M. Gran, *Torsion theories in homological categories*, J. Algebra 305 (2006) 18–47.

[2] S. E. Dickson, *A torsion theory for Abelian categories*, Trans. Amer. Math. Soc. 121 (1966) 223–235.

[3] D. Dikranjan and E. Giuli, *Closure operators. I*, Proceedings of the 8th international conference on categorical topology (L'Aquila, 1986), Topology Appl. 27 (1987) 129–143.

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