On Eraldo's and Bertrand's pairs of socks
— Some paradoxical observations —

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Dedicated to Eraldo Giuli and to Bertrand Russell on the occasion

of their 70-th and 138-th birthday respectively

> "*Since infinite, by definition, is incomprehensible, infinity, or the quality or attribute of being infinite, must be even more so, an abstraction of an unintelligibility.*"
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> — A Dictionary of Contemporary American Usage (1957), p. 245

The Setting

Mathematics is known as the science of the infinite. So, though each of us has only a finite number of pairs of socks, the concept of a countable* number of pairs of socks provides no difficulties to us. Simple, as it seems, it leads however to some astounding and amusing paradoxes. Even more surprisingly, the kind of paradoxes that arise depend heavily on the set theory used. In particular AC, the axiom of choice, and a strong negation of AC lead to rather different observations, some even contradicting each other. So, though each is true in its proper setting, there is no reasonable setting in which all of them are true simultaneously.

For this special occasion let us individualize our observations and let us assume that

(1) Eraldo has countably many pairs of socks and — being a resourceful Italian — he is able to simultaneously select one sock out of each pair.

(2) Bertrand has countably many pairs of socks too, but — being a sceptical Briton — he considers it impossible to simultaneously select one sock out of any given infinite number of pairs, because there exists no "*rule of choice*".

Since we all know Eraldo well enough (and are used to the axiom of choice) the paradoxes concerning Eraldo's socks are so familiar to us that many of us may have lost the feeling for their paradoxical nature. However it turns out that some of the statements about Bertrand's socks — though others appear rather bizarre — are in

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*Throughout this talk "*countable*" will mean "*countable and infinite*".

better harmony with our finitary experience than the corresponding statements about Eraldo's socks.

The Number of Socks

If we have n pairs of socks, then we have precisely $2n$ socks and (for $n > 0$) the number of our socks is larger than the number of our pairs of socks. However:

(E1) Eraldo has exactly as many socks as he has pairs of socks.

(B1) The number of Bertrand's socks is unknown to us. However we know that it is neither equal to nor larger nor smaller than the number of his pairs of socks. Both numbers are incomparable.

Even worse: It is impossible to count Bertrand's socks, i.e. to well-order them. In fact, a set of Bertrand's socks can be counted if and only if it is finite.

Even or Odd ?

If we have a finite number of pairs of socks, then the number of our socks will be even, but not odd. However:

(E2) The number of Eraldo's socks is simultaneously even and odd.

(B2) The number of Bertrand's socks fails to be odd. It might be even or might fail to be so.

Square ?

If we have a finite number of pairs of socks, then the number of socks could be square or could fail to be so. However:

(E3) The number of Eraldo's socks is square.

(B3) The number of Russell's socks fails to be square.

Even stranger than that: It is impossible to express the number of Russell's socks as a product of two infinite numbers. Even weirder: It is conceivable that the number of Russell's socks is a prime, i.e. divisible only by 1 and itself.

Removal of some Pairs of Socks

If we remove a pair of socks from a finite number of pairs, the removal can be detected by counting: The number of socks will have decreased properly. However:

(E4) If Max removes a pair from Eraldo's pairs of socks, then the number of Eraldo's socks remains unchanged.

Even stranger: Max may carefully remove countably many of Eraldo's pairs of socks without reducing the number of Eraldo's socks. Even weirder: Max may supply a countable number of his friends each with a countable number of Eraldo's pairs of socks without reducing the number of Eraldo's socks.

(B4) The removal of a single pair of Bertrand's socks properly reduces the number of Bertrand's socks.

Socks in the Washing Machine

If we wash n pairs of socks in a washing machine and afterwards rearrange the socks arbitrarily into pairs, then we will end up again with exactly n pairs of socks. However:

(E5) If we rearrange Eraldo's socks arbitrarily into pairs we will end up with as many pairs as we started with, namely countably many.

(B5) If we rearrange Bertrand's socks into pairs we may end up with countably many pairs, but it may also happen that we end up with an uncountable number of pairs.

Even stranger: It could happen that, by washing and rearranging the socks a countable number of times, we may at each time end up with a greater number of socks than at the preceding time. Even weirder: This rearranging procedure could be extended to any countable ordinal with ever increasing numbers of pairs.

Triples of Socks

Experts sometimes advise to keep our socks in triples instead of pairs, since the loss of one member of a triple will not affect the usefulness of the remaining pair. So let us try to rearrange our socks into triples. In the case of n pairs of socks this will be possible if and only if the number of socks is divisible by 3, say $2n = 3m$, and in this case we obtain exactly m triples. Moreover $m < n < 2n$. However:

(E6) Eraldo can rearrange his socks into triples, and the number of his socks, the number of his pairs of sock, and the number of his triples of socks will coincide.

(B6) Bertrand's socks can be rearranged into triples if and only if their number a is divisible by 3, say $a = 3b$, and in this case (which may or may not happen) he will obtain exactly b triples, where $b < a$, but b and the number of Bertrand's original pairs will be incomparable.

Stranger even is the fact that Bertrand — in spite of his sceptical attitude concerning the absence of rules — will be able to simultaneously select one member of

each triple, and afterwards simultaneously select one member of the remaining pair. Just find some suitable "rule of choice"!

Conclusion

Summing things up let me express my hope that I have been able to entertain and confuse you thoroughly. Keep in mind however the following slogan by Willard Van Orman Quine.

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"Irrefragability, thy name is mathematics."

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