# Secant Varieties of Segre Varieties and Tensor Decomposition

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Topics in Tensors.
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# Organization of the Talk

0) Some algebraic and geometric preliminaries.
1) What are Secant Varieties?
2) What are Segre Varieties?
3) What are the Secant Varieties of the Segre Varieties?
4) What are the dimensions of the Secant Varieties of the Segre Varieties?
    - Two factors;
    - More than two factors.
5) What is our approach?
6) What have we proved? What do we think is true?

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7) Why should you be interested in the answers to any of the questions above??

$V$ a vector space of dimension $n + 1$ over the field $k$.

$$\mathbb{P}^n_k = \mathbb{P}_k(V)$$

the projective space based on this vector space. I.e. the elements of this projective space are the one dimensional subspaces of $V$. (often we drop the "k")

$v \neq 0$, $v \in V$ then the element of $\mathbb{P}(V)$ defined by $v$ is denoted $[v]$. Clearly $[v] = [cv]$ for any $c \neq 0$, $c \in k$.

If we choose a basis $\{v_0, \dots, v_n\}$ of $V$ then if $v = \sum_{i=0}^{n} a_i v_i$ then we write $[v] = [a_0 : \dots : a_n]$. The "$a_i$" are the *projective coordinates* of the point $[v]$ with respect to that basis.

If $W \subset V$ is a subspace we have $\mathbb{P}(W)$ naturally contained in $\mathbb{P}(V)$ and call it a *(projective) linear subspace of* $\mathbb{P}(V)$.

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Consider the polynomial ring

$$R = k[x_0, \ldots, x_n].$$

If we ask: which polynomials vanish at a point in $\mathbb{P}^n$?, we are led to consider homogeneous polynomials and homogeneous ideals in the polynomial ring.

We define *projective algebraic sets* as those subsets of $\mathbb{P}^n$ which are all the common zeros of an ideal generated by homogeneous polynomials in $R$.

From now on $\mathbb{X} \subset \mathbb{P}^n$ is a *reduced*, *irreducible*, and *non-degenerate* projective variety, i.e. $\mathbb{X}$ arises as the set of zeroes of a prime homogeneous ideal which does not contain any linear forms (i.e. the set $\mathbb{X}$ does not lie in any projective linear subspace of $\mathbb{P}^n$).

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# Description of Secants

Let $z_0, \dots, z_r$ be *linearly independent* points in $\mathbb{P}^n$. Then

$$\mathbb{P}^r \simeq \langle z_0, \dots, z_r \rangle := J(z_0. \dots, z_r)$$

is called the *join* of the points $z_0, \dots, z_r$.

More generally, if $\mathbb{X}_0, \dots, \mathbb{X}_r$ ($r \leq n$) are a collection of non-degenerate irreducible varieties in $\mathbb{P}^n$, then

$$\overline{\bigcup \{J(z_0, \dots, z_r) \mid z_i \in \mathbb{X}_i, \{z_0, \dots, z_r\} \text{ lin. ind.}\}}$$

is called the *join of* $\mathbb{X}_0, \dots, \mathbb{X}_r$ and denoted

$$J(\mathbb{X}_0, \dots, \mathbb{X}_r).$$

If $\mathbb{X}_0 = \mathbb{X}_1 = \dots = \mathbb{X}_r = \mathbb{X}$ then

$$J(\mathbb{X}_0, \dots, \mathbb{X}_r) := Sec_r(\mathbb{X})$$

is the $r^{th}$ *secant variety of* $\mathbb{X}$.

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# Description of Secants

When $r = 1$ we refer to $Sec_r(\mathbb{X})$ as the *secant line variety*, when $r = 2$ the *secant plane variety* and so on.

An important and (usually) difficult question to answer is:

What is $\dim(J(\mathbb{X}_0, \ldots, \mathbb{X}_r))$ ?

and, more specifically,

What is $\dim Sec_r(\mathbb{X})$?

There is always a guess! but the guess is not always correct. We use a *parameter count*.

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To guess at the dimension of $Sec_r(\mathbb{X})$, when the dimension of $\mathbb{X}$ is $d$, we reason as follows: we need to pick $r+1$ points on $\mathbb{X}$, i.e. a point in $\mathbb{X}^{r+1}$, so that is a choice in a space of dimension $(r+1)\dim\mathbb{X}$.

Once having done that, we get a $\mathbb{P}^r$ and can choose any point in it, i.e. an additional $r$ dimensions.

On the other hand, if our variety $\mathbb{X}$ is in $\mathbb{P}^N$ for some $N$, then the dimension of the secant variety obviously cannot exceed $N$.

So, we get:

**the expected dimension of $Sec_r(\mathbb{X})$.**

If the dimension of $\mathbb{X}\subset\mathbb{P}^N$ is $d$, the expected dimension of $Sec_r(\mathbb{X})$ is

$$\min\{N, d(r+1)+r = r(d+1)+d\}.$$

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So, the *expected dimension* of the secant line variety of a variety of dimension $d$ should be

$$1(d+1)+d=2d+1.$$

So, for a curve (not in the plane), it should have dimension 3, for a surface (not in $\mathbb{P}^4$), it should have dimension 5.

The dimension cannot be larger than the *expected dimension* although it can be smaller. If, for some $Sec_r(\mathbb{X})$ the dimension is smaller than expected we call the (positive) difference between the two numbers the *defectivity of $\mathbb{X}$ for $r$-secants*, or simply say that the variety $\mathbb{X}$ is *defective*.

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# Example of a defective variety

Let $R = \mathbb{C}[x, y, z] = \oplus_{i=0}^{\infty} R_i$, where $R_i$ is the homogeneous piece of $R$ of degree $i$. It is the subvectorspace of $R$ generated by the monomials of degree $i$.

Consider the map

$$\phi : \mathbb{P}(R_1) = \mathbb{P}^2 \to \mathbb{P}(R_2) = \mathbb{P}^5$$

defined by

$$\phi([L]) = [L^2].$$

I.e. the image, $\mathbb{X}$, is a surface isomorphic to $\mathbb{P}^2$, and consists of all the quadratic forms (in three variables) which are the square of a linear form. I.e. the classes of symmetric $3 \times 3$ matrices of rank 1.

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# Example of a defective variety

The secant line variety of the surface $\mathbb{X}$, if not defective, should have dimension 5, i.e. should be all of $\mathbb{P}(R_2)$.

But, the secant line variety consists of the classes of quadratic forms of the form

$$[L_1^2 + L_2^2],$$

and those correspond to symmetric $3 \times 3$ matrices of rank 2.

However, all of these lie on the hypersurface obtained by considering the determinant of the generic $3 \times 3$ symmetric matrix. This is a hypersurface in $\mathbb{P}^5$ of degree 3, i.e. something of dimension 4, not 5.

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How can we deal with this problem of the dimension of the secant varieties. The (justly) famous *Terracini Lemma* give us a hint for finding that dimension.

**Terracini's Lemma:**

Let $P$ be a generic point of $\mathbb{X} = J(\mathbb{X}_0, \dots, \mathbb{X}_r)$.
(So $P \in J(z_0, \dots z_r)$ for linearly independent points $z_i \in \mathbb{X}_i$).
If $T_{P,\mathbb{X}}$ is the (projectivized) tangent space to $\mathbb{X}$ at $P$, then

$$T_{P,\mathbb{X}} = \langle T_{z_0,\mathbb{X}_0}, \dots, T_{z_r,\mathbb{X}_r} \rangle$$

and so

$$\dim \mathbb{X} = \dim \langle T_{z_0,\mathbb{X}_0}, \dots, T_{z_r,\mathbb{X}_r} \rangle.$$

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# Segre Varieties (two factors)

Let $V$ and $W$ be two vector spaces over $k$ of dimensions $(n+1)$ and $(m+1)$ respectively. Then $V \otimes W$ is a vector space of dimension $(n+1)(m+1)$.

The *Segre Variety* is the image of the mapping

$$\mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow \mathbb{P}(V \otimes W)$$

defined by

$$[v] \times [w] \longrightarrow [v \otimes w].$$

Roughly speaking, the image is the collection of all the decomposable tensors in $V \otimes W$ and we have identified (the map is an injection) the product of the two projective spaces $(\mathbb{P}(V) \times \mathbb{P}(W))$ with the set of decomposable tensors.

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It is illuminating to write down the mapping above in coordinates. If we use $\{v_0, \ldots, v_n\}$ as a basis for $V$, $\{w_0, \ldots, w_m\}$ as a basis for $W$ and $\{v_i \otimes w_j\}$ as a basis for $V \otimes W$, then the map becomes

$$[a_0 : \ldots : a_n] \times [b_0 : \ldots : b_m] \longrightarrow [\ldots : a_i b_j : \ldots]$$

which becomes even more interesting if we write it as a matrix product

$$\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \times [b_0 \ldots b_m] = \begin{bmatrix} - & \ldots & - \\ - & a_i b_j & - \\ - & \ldots & - \end{bmatrix}$$

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Thus, we see that the decomposable vectors in $V \otimes W$ can be identified with the $(n+1) \times (m+1)$ matrices of rank 1.

So, if we let $\mathbb{X} = \mathbb{P}(V) \times \mathbb{P}(W) \subset \mathbb{P}^{(n+1)(m+1)-1}$ then we see that the general points on $Sec_r(\mathbb{X})$ are the sums of $r+1$ matrices of rank 1, i.e. are the sums of $r+1$ decomposable tensors.

**Example:** Let's look at the example of $\mathbb{P}^1 \times \mathbb{P}^1$. The Segre embedding of this into $\mathbb{P}^3$ is given by

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \times \begin{bmatrix} b_0 & b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} a_0 b_0 & a_0 b_1 \\ a_1 b_0 & a_1 b_1 \end{bmatrix}$$

and

$$Sec_1(\mathbb{P}^1 \times \mathbb{P}^1) =$$

$$\{2 \times 2 \text{ matrices, sum of 2 matrices of rank 1}\}$$

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However, we have the following theorem from linear algebra

**Theorem:** A matrix has rank $\leq r$ if and only if it is the sum of $r$ matrices of rank 1.

Since every $2 \times 2$ matrix has rank $\leq 2$, every $2 \times 2$ matrix is a sum of $\leq 2$ matrices of rank 1. I.e. every tensor in $k^2 \otimes k^2$ is the sum of $\leq 2$ decomposable tensors.

Note that in this case, dim $Sec_1(\mathbb{P}^1 \times \mathbb{P}^1) = 3$ which is the expected dimension $(\min\{5, 3\})$.

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**Example:** $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \times \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_0 b_0 & a_0 b_1 & a_0 b_2 \\ a_1 b_0 & a_1 b_1 & a_1 b_2 \\ a_2 b_0 & a_2 b_1 & a_2 b_2 \end{bmatrix}$$

In this case,

$$\text{expected dim } Sec_1(\mathbb{P}^2 \times \mathbb{P}^2) = \min\{4 + 4 + 1, 8\} = 8.$$

A $3 \times 3$ matrix has rank $\leq 2$ if and only if the determinant of that matrix is 0. I.e all the tensors in $k^3 \otimes k^3$ which are the sum of $\leq 2$ decomposable tensors, correspond to matrices with determinant 0.

But the determinant of a general $3 \times 3$ matrix is a homogeneous polynomial of degree 3 whose zeroes are a hypersurface in $\mathbb{P}^8$, i.e. something of dimension 7, not dimension 8.
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This sort of thing happens all the time, but the case for Segre varieties with two factors is very well understood since all of the theory is in terms of ranks of matrices and that is understood very well from both an algebraic and geometric standpoint.

# Segre Varieties with more than 2 factors

Now let $V_i$, $i = 1, \ldots, t$ be vector spaces of dimension $n_i + 1$ and let a basis for $V_i$ be given by $\{e_{i0}, \ldots, e_{in_i}\}$.

In this case, the Segre Variety,

$$\mathbb{X} = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t)$$

is contained in

$$\mathbb{P}(V_1 \otimes \cdots \otimes V_t) = \mathbb{P}^N$$

where $N = [\Pi_{i=1}^{t}(n_i + 1)] - 1$.

The Segre map is the obvious extension of the map we gave for two factors and it is easy to see that the image of $\mathbb{X}$ in $\mathbb{P}^N$ is precisely the classes of the decomposable tensors.

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**Definition:** We say that $[u] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ has *tensor rank equal to* $(r+1)$ if

$$u = u_0 + \cdots + u_r$$

where the $u_i$ are decomposable tensors, and there is no shorter such decomposition.
Call this set of tensors $U_{r+1}$.

Since $V_1 \otimes \cdots \otimes V_t$ has a basis of decomposable tensors, such an $r$ always exists.
Clearly, with $\mathbb{X}$ as above,

$$U_{r+1} \subset Sec_r(\mathbb{X}).$$

The problem is that, unlike the case of two factors, $Sec_r(\mathbb{X})$ can contain tensors of very high tensor rank, since the secant variety is the *closure* of $U_{r+1}$.

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**Example:** Let me give you an example to show how things can go bad. It is a bit difficult to find a simple example for tensors but there is a lovely example for *symmetric tensors*.

Let $R = k[x,y] = \oplus_{i=0}^{\infty} R_i$. Let $V = k[x,y]_1$ (2-dimensional vector space) and $W = k[x,y]_3$ (4-dimensional). We define

$$\phi : \mathbb{P}(V) = \mathbb{P}^1 \longrightarrow \mathbb{P}(W) = \mathbb{P}^3, \phi([L]) = [L^3]$$

The image of this map is a curve the *rational normal curve* of $\mathbb{P}^3$. It's a curve of degree 3 (i.e. when you cut it with a general hyperplane of $\mathbb{P}^3$ you get three points).

Points of $\mathbb{P}^3$ which are on true secant lines have the form $[F] = [L_1^3 + L_2^3]$.

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With this problem in mind, another definition has been made.

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$$\textit{border rank} = (r+1)$$

if and only if $u \in Sec_r(\mathbb{X})$.

With this definition one can ask a somewhat simpler question.

What is the *maximum* border rank, $(r+1)$, of $[u] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ ?
(This number is sometimes called the *essential rank* of $V_1 \otimes \cdots \otimes V_t$.)

Equivalently: what is the least integer $r$ so that $Sec_r(\mathbb{X}) = \mathbb{P}^N$.

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Equivalently: what is the least integer $r$ so that $Sec_r(\mathbb{X}) = \mathbb{P}^N$.

# A Word About Our Method:

Our method for computing the dimensions of the Secant Varieties for the Segre Varieties is essentially based on Terracini's Lemma.

Starting from that we build a homogeneous ideal in a polynomial ring with the property that the dimension of one of its graded pieces gives the dimension of the tangent space to the secant variety in question and hence the dimension of the secant variety.

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# A Word About Our Method:

In the particular case we are considering, i.e. when

$$\mathbb{X} \simeq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \subset \mathbb{P}^N$$

(where $N = [\Pi_{i=1}^{t}(n_1 + 1)] - 1$) and we are interested in finding the dimension of $Sec_r(\mathbb{X})$, then we associate to $\mathbb{X}$ a homogeneous ideal in the polynomial ring in

$$1 + n_1 + \cdots + n_t$$

variables, i.e. the polynomial ring associated to $\mathbb{P}^n$, $n = n_1 + \ldots + n_t$.

# A Word About Our Method:

That ideal has, in this case, a very simple form, namely

$$I = \Pi_1^{t-1} \cap \ldots \cap \Pi_t^{t-1} \cap \wp_0^2 \cap \ldots \cap \wp_r^2$$

where $\Pi_i$ is a coordinate linear space in $\mathbb{P}^n$ of dimension $n_i - 1$ and $\wp_i,\ i = 0, \ldots, r$, are ideals of $r + 1$ generic points in $\mathbb{P}^n$.

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# Results : positive results

As a small sample of some of the results we have found I will mention:

- All the secant varieties of

$$\mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \quad (r - \text{factors})$$

  have the expected dimension for $r \geq 5$.

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# Results: Defective Varieties

- Unbalanced. In this case,

$$\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^{n}$$

with $n_1 \leq \cdots \leq n_t$. If

$$\alpha = \Pi_{i=1}^{n}(n_i + 1) + 1 - \sum_{i=1}^{n} n_i \leq n$$

then $Sec_{s-1}(\mathbb{X})$ is deficient for $\alpha \leq s \leq n$.

- $$\mathbb{X} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^{26}$$

  is defective.
  In fact, $\dim Sec_3(\mathbb{X}) = 25$, a hypersurface, while the expected dimension is $4 \cdot 6 + 3 = 27$, hence should fill the space.

- $$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n = \mathbb{X}$$

  is defective.
  In fact $Sec_{2n}(\mathbb{X})$ should be a hypersurface in $\mathbb{P}^N$, $N = 4n^2 + 8n + 3$, but it actually lies on 2 hypersurfaces.

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# Conjecture: Abo-Ottaviani-Peterson

The defective Segre varieties above and:
$\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n$ ($n$ even) (Strassen);
$\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$ (Abo, Ottaviani, Peterson)
are the only defective Segre varieties with three or more factors.

# References

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