Bounds on the largest families of subsets with forbidden subposets

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Abstract

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, families $\mathcal{F}$ of its subsets will be investigated. An old theorem of Sperner (1928) says that if there is no inclusion ($F \in \mathcal{F}, G \in \mathcal{F}, F \neq G$ then $F \not\subset G$) then the largest family under this condition is the one containing all $\lfloor \frac{n}{2} \rfloor$-element subsets of $[n]$. Of course, Sperner's theorem can be formulated in terms of 0-1 matrices. Let the matrix have n columns, a row of the matrix is the characteristic vector of a subset of $[n]$. A family $\mathcal{F}$ is therefore a 0-1 matrix with n columns and $|\mathcal{F}|$ rows, where the order of the rows is irrelevant. Sperner's condition for this matrix is that one can find a column for any pair of rows so that the values at the two crossing points are 0 and 1, in this order. (Consequently there is another column with values 1 and 0.) Sperner's theorem maximizes the number of rows under this condition, if the number of columns is given.

This theorem has many consequences. It helps to find (among others) the maximum number of mimimal keys in a database, the maximum number of subsums of secret numerical data what can be released without telling any one of the data, bounds of the distribution of the sums $\sum \pm a_i$ for the vectors $a_i (1 \leq i \leq n)$.

We will consider its certain generalisations in the present lecture. They are useful in proving theorems in number theory. geometry, etc. Again, the maximum size of $\mathcal{F}$ is to be found under the condition that a certain configuration is excluded. The configuration here is always described by inclusions. More formally, let P be a poset. The maximum size of a family $\mathcal{F} \subset 2^{[n]}$ which does not contain P as a (non-necessarily induced) subposet is denoted by $\mathrm{La}(n, P)$.

If P consist of two comparable elements, then Sperner's theorem gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

In most cases, however $\mathrm{La}(n, P)$ is only asymptotically determined in the sense that the main term is the size of the largest level (sets of size $\lfloor \frac{n}{2} \rfloor$) while the second term is $\frac{c}{n}$ times the second largest level where the lower and upper estimates contain different constants c.

Let e.g. the poset N consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. In a relatively new paper the author jointly with J.R. Griggs determined $\mathrm{La}(n, N)$.

Theorem

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+\Omega(\frac{1}{n^2})\right) \le \mathrm{La}(n,N) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+O(\frac{1}{n^2})\right).$$

Similar results will be surveyed, also introducing a method.