INTERLACING PROPERTIES FOR HERMITIAN MATRICES WHOSE GRAPH IS A GIVEN TREE

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Abstract. We extend some interlacing properties of the eigenvalues of tridiagonal matrices to Hermitian matrices whose graph is a tree. We also give a graphical interpretation of the results. We use the work on matchings polynomials by O.L. Heilmann and E.H. Lieb.

1. Introduction

The study of behavior of eigenvalues of matrices of a special shape has been motivated by different fields of applied mathematics. Interest in spectra of tridiagonal matrices has increased in recent years in different branches such as numerical analysis, special functions, partial differential equations, and, naturally, linear algebra. Several fast and parallel algorithms for calculating these spectra have been designed (e.g. [1, 9, 13]).

Krishnakumar and Morf [13] established a divide-and-conquer approach for the solution of symmetric tridiagonal eigenvalue problems. This highly accurate method is based on a sequential doubling in the evaluation of the characteristic polynomial and a bisection method. In fact they consider partitioning a matrix and applying the three-term recurrence to the submatrices, yielding a recursion for which the order doubles at each step. A key feature of this approach is natural concurrency, which readily adapts to computing equipment with parallel architectures. In these circumstances, the computing time for all eigenvalues is \( O(n \log n) \) instead of \( O(n^2) \).

In 1992, Hill and Parlett [9] extended Cauchy’s interlacing theorem and other interlacing results for eigenvalues of tridiagonal matrices and related them to the last component of eigenvectors. Later on, Bar-On [1] generalized the main results of [9] giving new interlacing properties for eigenvalues of a tridiagonal symmetric matrix. Some relations relating the eigenvalues of a tridiagonal symmetric matrix to those of its leading and trailing submatrices have been proved. These theoretical results have been generalized to specially structured symmetric matrices, and applied to obtain fast and efficient parallel algorithms for locating the eigenvalues of matrices of very large order.

Kulkarni, Schmidt and Sze-Kai Tsui [12] considered the so-called tridiagonal pseudo-Toeplitz matrices, tridiagonal matrices which contain a Toeplitz matrix in the upper left block. They established a connection between the characteristic polynomials of these tridiagonal pseudo-Toeplitz matrices and the Chebyshev polynomials of the second kind, whereby one can locate the eigenvalues that fall in the intervals determined by the roots of some Chebyshev polynomials. They gave some special cases where the location of all eigenvalues can be determined. In fact, their results can be generalized to the broader class of Jacobi matrices.

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Our aim here is to unify some important theoretical results which deal with the spectra of tridiagonal matrices, generalize them to matrices whose graph is a given tree and bring the results together in one place. More precisely, we give the location of the eigenvalues of weighted trees in terms of spectra of an upper left block subtree and of a bottom right tridiagonal block. For this purpose, we derive some important identities for the characteristic polynomial of a weighted tree based on the work of Heilmann and Lieb [7], done in a different context. These authors, motivated by problems in quantum chemistry, established results which are the foundation of some recent developments on matchings polynomials (cf. [4, 5]). At the end we mention some results on the multiplicity of the eigenvalues of weighted trees. Some examples are given.

2. Weighted graphs

A weighted finite graph $G$ with $n$ vertices is a pair $(V, E)$, where $V$ is the vertex set $\langle n \rangle := \{1, \ldots, n\}$ and $E$ is a subset of $V \times V$; to each ordered pair $e = (i, j)$ we assign a complex number, the weight of $e$, and to $(j, i)$ its conjugate. We say that the vertices $i$ and $j$ are adjacent, and write $i \sim j$, if $(i, j)$ is an edge of $G$, with $i \neq j$. The symbol $\simeq$ means adjacent or equal. Throughout, we assume that the graph $G$ has loops, i.e., $(i, i)$ is an edge to which we assign a real number.

The (weighted adjacency) matrix of $G$ is the Hermitian matrix of order $n$ where the $(i, j)$-entry is the weight of the pair $(i, j)$.

Given an edge $e = (i, j)$ of $G$, $G\setminus e$ is obtained by deleting $e$ but not the vertex $i$ or $j$. In this case the matrix of $G\setminus e$ is equal to the one of $G$, except for the $(i, j)$-entry and, by symmetry, the $(j, i)$-entry, which are replaced by zero.

If $A = (a_{ij})$ is a Hermitian matrix of order $n$, the (weighted) graph of $A$, $G = G(A)$, is determined entirely by the off-diagonal entries of $A$: the vertex set of $G$ is $\langle n \rangle$ and $i$ and $j$ are adjacent if and only if $a_{ij} \neq 0$. If $A$ is a $0 - 1$ matrix, with main diagonal equal to zero, then $A$ is the adjacency matrix of $G(A)$. We define

$$\mathcal{H}(G) = \{ A \mid A = A^*, G(A) = G \} ,$$

the set of all Hermitian matrices that share a common graph $G$. Since our study is dedicated to the eigenvalues of Hermitian matrices, we will also consider

$$\tilde{\mathcal{H}}(G) = \{ DAD^{-1} \mid A \in \mathcal{H}(G), \ D \text{ is a diagonal matrix} \} ,$$

i.e., the set of all matrices that are diagonally similar to Hermitian matrices whose graph is $G$.

A tree is a connected graph without cycles, and a (disconnected) forest is a graph of which each component is a tree.

3. The characteristic polynomial of a weighted tree

There are close connections between the theory of matchings in graphs and the characteristic polynomial of an adjacency matrix of a graph. Recall that the matchings polynomial of a graph $G$ is $\mu(G, x) := \sum_k (-1)^k p(G, k) x^{n-2k}$, where $p(G, k)$ is the number of elements of a set of $k$ disjoint edges, no two of which have a vertex in common.

Though the matchings polynomial of a graph has many interesting properties, the task of computing this polynomial for a given graph is hard. In general there is no easy way of computing $\mu(G, x)$. Thus the matchings polynomial is in this regard a more intractable object than the characteristic polynomial. Nonetheless, it is known that $G$ is a forest if and only if both polynomials coincide and there are simple recurrences that enable us to compute the matchings polynomials of small graphs with some facility. For example, the
matchings polynomials of bipartite graphs are essentially the same as “rook polynomials” (cf. [4]). An unexpected property of matchings polynomials is that all their zeros are real. In the paper [7] we can find three distinct proofs of this fact. Therefore if $G$ is a tree, then all the eigenvalues of the adjacency matrix of $G$ are real. Here we strengthen this result.

Given a graph $G$, the characteristic polynomial of $A(G)$,

$$\varphi(G, x) = \det(xI - A(G)),$$

is sometimes called the characteristic polynomial of $G$ and simply denoted by $\varphi(G)$.

**Lemma 3.1.** Let $F$ be a forest with components $T_1, \ldots, T_\ell$, then $\varphi(F) = \varphi(T_1) \cdots \varphi(T_\ell)$.

Let us define $w_{ij}(A) = |a_{ij}|^2$ if $i \neq j$ and, otherwise, $w_{ii}(A) = a_{ii}$. Sometimes we abbreviate $w_{ij}(A)$ to $w_{ij}$. The next result provides a general recurrence relation between different characteristic polynomials.

**Lemma 3.2.** If $e = (i, j)$ is an edge in a (weighted) tree $T$, then

$$(3.1) \quad \varphi(T, x) = \varphi(T \setminus e, x) - w_{ij}\varphi(T \setminus ij, x).$$

**Proof.** Let $E_{ij}$ be the matrix with $ij$-entry equal to 1, and all other entries equal to zero. Denote by $E$ the sum $a_{ij}E_{ij} + \bar{a}_{ij}E_{ji}$. Notice that

$$A(T) = A(T \setminus e) + E.$$

Since the determinant is a multilinear function on the columns and $T$ is a tree, we get (3.1). □

**Theorem 3.3.** Let $i$ be a vertex of a weighted tree $T$. Then

$$(3.2) \quad \varphi(T, x) = (x - w_{ii})\varphi(T \setminus i, x) - \sum_{k \sim i} w_{ki}\varphi(T \setminus ki, x).$$

**Proof.** The equality (3.2) can be derived by iterating the formula (3.1). □

A routine induction argument, based on (3.2), gives us an expression for the derivative of the characteristic polynomial.

**Corollary 3.4.** If $T$ is a weighted tree, then

$$\varphi'(T, x) = \sum_{i \in V(T)} \varphi(T \setminus i, x).$$

We now state a result which we will use often. The general interlacing theorem between the eigenvalues of a Hermitian matrix and one of its principal submatrices is well known in the literature (e.g. [10]). It follows from the Interlacing Theorem:

**Theorem 3.5.** Let $T$ be a tree on $n$ vertices and $A \in \mathcal{H}(T)$. Then all eigenvalues of $A(T)$ are real, say

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Furthermore, if $i$ is a vertex in $V(T)$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ are the eigenvalues of $A(T \setminus i)$, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n,$$

i.e., the eigenvalues of $A(T)$ interlace those of $A(T \setminus i)$.

Theorem 3.5 has a well known corollary for tridiagonal matrices already proved elsewhere (cf. [12]).

**Corollary 3.6.** Let $P$ be a path on $n$ vertices and $A \in \mathcal{H}(P)$. Then $A$ has $n$ distinct real eigenvalues.
4. Christoffel-Darboux identity

As we already pointed out, the work of Heilmann and Lieb [7] has had some important implications in the theory of matchings polynomials with a close connection to orthogonal polynomials (for more details see e.g. [2]). Here we derive some important identities for characteristic polynomials of matrices in the set $\mathcal{H}(T)$.

Let $P_{ij}$ denote the (unique) path in a tree joining vertex $i$ to $j$. Given a path $P$ in $T$ with more than one vertex, let us define $W(P) = \prod_{(i,j)} w_{ij}(P)$, where the product is taken over the edges $(i,j)$ of $P$, with $i < j$.

**Theorem 4.1** (Christoffel-Darboux Identity). Let $T$ be a weighted tree on $n$ vertices. For every vertex $i \in V(T)$,

$$
\varphi(T, x) \varphi(T \backslash i, y) - \varphi(T, y) \varphi(T \backslash i, x) = (x - y) \sum_{j=1}^{n} W(P_{ij}) \varphi(T \backslash P_{ij}, x) \varphi(T \backslash P_{ij}, y).
$$

**Proof.** For trees with one or two vertices the result is trivial. From (3.2), we have the equations
\[
x \varphi(T \backslash i, x) \varphi(T \backslash i, y) = \varphi(T, x) \varphi(T \backslash i, y) + \sum_{k \not= i} w_{ki} \varphi(T \backslash ki, x) \varphi(T \backslash i, y)
\]
and
\[
y \varphi(T \backslash i, y) \varphi(T \backslash i, x) = \varphi(T, y) \varphi(T \backslash i, x) + \sum_{k \not= i} w_{ki} \varphi(T \backslash ki, y) \varphi(T \backslash i, x).
\]
Subtracting the second equation from the first we get
\[(x - y) \varphi(T \backslash i, y) \varphi(T \backslash i, x) = \varphi(T, x) \varphi(T \backslash i, y) - \varphi(T, y) \varphi(T \backslash i, x) - \sum_{k \not= i} w_{ki} [\varphi(T \backslash i, x) \varphi(T \backslash ki, y) - \varphi(T \backslash i, y) \varphi(T \backslash ki, x)].\]

Applying the hypothesis on $\varphi(T \backslash i, x) \varphi(T \backslash ki, y) - \varphi(T \backslash i, y) \varphi(T \backslash ki, x)$, with the convention $\varphi(0, x) = 1$, we get the result. \hfill $\square$

**Corollary 4.2.** Let $T$ be a weighted tree on $n$ vertices. For every vertex $i \in V(T)$,

$$
\varphi'(T, x) \varphi(T \backslash i, x) - \varphi(T, x) \varphi'(T \backslash i, x) = \sum_{j=1}^{n} W(P_{ij}) \varphi(T \backslash P_{ij}, x)^2.
$$

**Proof.** Letting $y \to x$ in (4.1), we get (4.2) since we may write
\[
\varphi(T, x) \varphi(T \backslash i, y) - \varphi(T, y) \varphi(T \backslash i, x) = [\varphi(T, x) - \varphi(T, y)] \varphi(T \backslash i, y) - [\varphi(T \backslash i, x) - \varphi(T \backslash i, y)] \varphi(T, x).
\]

If we consider the sum over all vertices of $T$ in (4.2), then from Corollary 3.4 we get:

**Corollary 4.3.**

$$
\varphi'(T, x)^2 - \varphi''(T, x) \varphi(T, x) = \sum_{i,j=1}^{n} W(P_{ij}) \varphi(T \backslash P_{ij}, x)^2.
$$

**Theorem 4.4.** Let $T$ be a weighted tree on $n$ vertices. For every distinct pair $i, j \in V(T)$,

$$
\varphi(T \backslash i, x) \varphi(T \backslash j, x) - \varphi(T \backslash ij, x) \varphi(T, x) = W(P_{ij}) \varphi(T \backslash P_{ij}, x)^2.
$$
Proof. Once again the cases \( n = 1 \) and \( n = 2 \) are trivial. We use induction on \( n \) as in the propositions before. For that we only have to consider (3.2) and
\[
\varphi(T \setminus j, x) = (x - w_i)\varphi(T \setminus ij, x) - \sum_{k \sim i} w_{ki}\varphi(T \setminus kij, x).
\]
\[\square\]

5. Location of the eigenvalues of weighted trees

When we consider the Jacobi matrix
\[
J_{n+1} := \begin{pmatrix}
\alpha_0 & \beta_0 \\
\gamma_1 & \alpha_1 & \beta_1 \\
\gamma_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\gamma_n & \alpha_n \\
\end{pmatrix},
\]
its off-diagonal entries enter into its characteristic equation only as products \( \beta_i \gamma_{i+1} \). Thus it suffices to consider only matrices that have superdiagonal entries normalized to 1, i.e.,
\[
\tilde{J}_{n+1} := \begin{pmatrix}
\alpha_0 & 1 \\
\gamma_1 & \alpha_1 & 1 \\
\gamma_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\gamma_n & \alpha_n \\
\end{pmatrix}.
\]
From Corollary 3.6 we conclude the well-known result that \( J_n \) and, therefore, \( \tilde{J}_n \), with \( \gamma_i > 0 \), have \( n \) distinct real eigenvalues. Note that \( \tilde{J}_n \in \hat{H}(P) \), where \( P \) is a path on \( n \) vertices. In particular, when we consider tridiagonal Toeplitz matrices, i.e., making \( \alpha_i \)’s and \( \gamma_i \)’s all equal to \( \alpha \) and \( \gamma \), respectively, the characteristic polynomial of \( J_n \) is
\[
p_n(x) = (\sqrt{\gamma})^n U_n \left( \frac{x - \alpha}{\sqrt{\gamma}} \right),
\]
where \( U_n \) is the Chebyshev polynomial of second kind of degree \( n \), which satisfy the three-term recurrence relations
\[
2xU_n(x) = U_{n+1}(x) + U_{n-1}(x),
\]
for all \( n = 1, 2, \ldots \), with initial conditions \( U_0(x) = 1 \) and \( U_1(x) = 2x \). It is well known (cf. [2], e.g.) that each \( U_n \) also satisfies
\[
U_n(x) = \frac{\sin(n + 1)\theta}{\sin \theta}, \quad x = \cos \theta \quad (0 \leq \theta < \pi)
\]
for all \( n = 0, 1, 2 \ldots \), from which one easily deduce the orthogonality relations
\[
\int_{-1}^1 U_n(x)U_m(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2} \delta_{n,m}.
\]

Lemma 5.1. The eigenvalues of the tridiagonal Toeplitz matrix defined in (5.1), with \( \alpha_i = \alpha \) and \( \gamma_i = \gamma \), are
\[
\lambda_\ell = \alpha - 2\sqrt{\gamma} \cos \left( \frac{\ell \pi}{n+1} \right),
\]
for \( \ell = 1, 2, \ldots, n \).
We remark that if $\gamma < 0$, then $p_n(x) = (i\sqrt{-\gamma})^n U_n \left( \frac{x-a}{2\sqrt{-\gamma}} \right)$, where $i = \sqrt{-1}$.

Given a weighted tree $T$ on $m + n$ vertices, let $A^{m,n}$ be the matrix diagonally similar to the Hermitian matrix $A(T)$ partitioned in the following way

\[
A^{m,n} = \begin{pmatrix}
A(T^m) & 1 \\
\gamma_1 & \alpha_1 & 1 \\
\gamma_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\gamma_n & \alpha_n & 1
\end{pmatrix},
\]

where $T^m$ is a subtree of $T$. We assume that the tridiagonal right block is unreduced, with $\gamma_i > 0$. Using the Laplace expansion we get

\[
\varphi(T, x) = \varphi(T^m, x)\varphi(P^n, x) - \gamma_1\varphi(T^m \setminus m, x)\varphi(P^n \setminus m + 1, x),
\]

where $P^n$ is the path in $T$ on the $n$ vertices $\{m+1, \ldots, m+n\}$. Suppose $\theta$ is an eigenvalue of $A^{m,n}$. If $\theta$ is not a common zero of $\varphi(T^m \setminus m, x)$ and $\varphi(P^n, x)$, then

\[
\frac{\varphi(T^m, \theta)}{\varphi(T^m \setminus m, \theta)} = \gamma_1 \frac{\varphi(P^n \setminus m + 1, \theta)}{\varphi(P^n, \theta)}.
\]

Notice that given any weighted tree, if we define

\[
p(x) := \frac{\varphi(T, x)}{\varphi(T^i, x)},
\]

then

\[
p'(x) := \frac{\varphi'(T, x)\varphi(T^i, x) - \varphi(T, x)\varphi'(T^i, x)}{(\varphi(T^i, x))^2},
\]

which is positive by (4.2), i.e., $p(x)$ is strictly increasing. If we denote

\[
f(x) = \frac{\varphi(T^m, x)}{\varphi(T^m \setminus m, x)} \quad \text{and} \quad g(x) = \gamma_1 \frac{\varphi(P^n \setminus m + 1, x)}{\varphi(P^n, x)},
\]

then $f(x)$ is strictly increasing and $g(x)$ is strictly decreasing, in each interval where they are defined.

A similar analysis can be easily done if instead of the right Jacobi block in (5.2), we consider $A(T^n)$, for a more general tree $T^n$ on $n$ vertices.

Consider the extended sets of eigenvalues of $A(T^m \setminus m)$,

\[
\eta = \{-\infty = \eta_0 < \eta_1 \leq \cdots \leq \eta_{m-1} < \eta_m = +\infty\}
\]

and of $A(T^m)$, $\xi = \{-\infty = \xi_0 < \xi_1 \leq \cdots \leq \xi_m < \xi_{m+1} = +\infty\}$. Similarly we consider $\zeta = \{-\infty = \zeta_0 < \zeta_1 < \cdots < \zeta_n < \zeta_{n+1} = +\infty\}$ the extended set of eigenvalues of $A(P^n)$ and $\rho = \{-\infty = \rho_0 < \rho_1 < \cdots < \rho_{n-1} < \rho_n = +\infty\}$ of $A(P^n \setminus m + 1)$. By Theorem 3.5

\[-\infty < \xi_1 \leq \eta_1 \leq \cdots \leq \eta_{m-2} \leq \xi_{m-1} \leq \eta_{m-1} \leq \xi_m < +\infty
\]

and

\[-\infty < \zeta_1 < \rho_1 < \cdots < \rho_{n-2} < \zeta_{n-1} < \rho_{n-1} < \zeta_n < +\infty.
\]

Since $f(x)$ is strictly increasing and $g(x)$ is strictly decreasing, from the intermediate value theorem, they intersect each other at least once in every interval $[\xi_\ell, \xi_{\ell+1}]$, for $0 \leq \ell \leq m$. If there are no poles and no zeros of $g(x)$ lying in $[\xi_\ell, \xi_{\ell+1}]$, then $f(x)$ and $g(x)$ intersect
exactly once in this interval. If there exists a zero, say \( \rho_r \), then \( f(x) \) and \( g(x) \) agree exactly once in each interval \([\xi_\ell, \rho_r]\) and \((\rho_r, \xi_{\ell+1}]\).

Suppose now that some pole of \( g(x) \), say \( \zeta_r \), is in \((\xi_\ell, \xi_{\ell+1}]\) and a zero \( \rho_r \) of \( g(x) \) in \((\xi_\ell, \zeta_r)\). Then \( f(x) \) and \( g(x) \) agree exactly once in each interval \([\xi_\ell, \rho_r]\) and \((\rho_r, \xi_{\ell+1}]\).

More precisely, in each interval \([\xi_\ell, \rho_r]\) and \((\zeta_r, \xi_{\ell+1}]\). If there exists another zero of \( g(x) \) in the interval, then there is one zero of \( \varphi(T, x) \) in the interval \((\rho_r, \rho_{r+1}]\) or, effectively, in the interval \((\zeta_r, \rho_{r+1}]\).

A similar analysis applies if the zeros \( \rho_r < \cdots < \rho_{r+k} \) lie in the interval \((\xi_\ell, \xi_{\ell+1}]\). We have just proved the following proposition:

**Theorem 5.2.** Let \( A^{m,n} \) be the Hermitian matrix defined in \((5.2)\). Let the extended set \( \pi \) be the union of \( \xi \) and \( \rho \). Then in each interval \([\pi_\ell, \pi_{\ell+1}]\), for \( \ell = 0, \ldots, m+n \), there is exactly one distinct eigenvalue of \( A^{m,n} \).

Note that, if \( \pi_k = \pi_{k+1} \), for some \( k \), then the interval \([\pi_k, \pi_{k+1}]\) reduces to a single point, namely \( \pi_k \).

An analogous result on the location of the eigenvalues of \( A^{m,n} \) is obtained if \( \pi \) is the union of \( \eta \) and \( \zeta \). We can reformulate Theorem 5.2 in the same way as in \([12]\).

**Theorem 5.3.** For \( \ell = 0, \ldots, m \), each interval \([\xi_\ell, \xi_{\ell+1}]\) contains one more root of \( \varphi(T, x) \) than zeros of \( g(x) \).

Theorem 5.2 is a generalization of the main results of \([1]\). For example, if \( A(T^m) \) is an unreduced tridiagonal matrix, then in each interval \((\pi_\ell, \pi_{\ell+1}]\), for \( \ell = 0, \ldots, m+n \), there is exactly one different eigenvalue of \( A^{m,n} \). We can get similar results, after some manipulations, from \([19]\).

### 6. Examples

In this section we present some examples. First let us put \( n = 1 \) in \((5.2)\), i.e., consider

\[
A^{m,1} = \begin{pmatrix} \end{pmatrix}
\[
A(T^m) \quad 1
\] \begin{pmatrix} \end{pmatrix}
\]

Using the analysis described in Theorem 5.2, we can state:

**Theorem 6.1.** If \( \eta_\ell < \alpha_1 < \xi_{\ell+1} \), for some \( \ell \), then there is precisely one root of \((5.3)\), lying in the interval \((\eta_\ell, \alpha_1)\), and \((\alpha_1, \xi_{\ell+1}]\) contains none. Similarly, if \( \xi_{\ell+1} < \alpha_1 < \eta_{\ell+1} \), for some \( \ell \), then there is precisely one root of \((5.3)\), lying in the interval \((\alpha_1, \eta_{\ell+1})\) and \((\xi_{\ell+1}, \alpha_1)\) contains none.

In fact, this result generalizes some other examples given, for example, in \([9, 12]\).

Consider now the matrix:

\[
A = \begin{pmatrix} 1 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 0 & 0 & 0 \\
-i & 1 & -1 & 1 & -\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{2} & 2 & -1+i & 0 \\
0 & 0 & 0 & 0 & -1-i & -1 & -2 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}.
\]

The graph of \( A \) is
The eigenvalues of
\[
\begin{pmatrix}
1 & 0 & 0 & i \\
0 & 0 & 0 & 1 \\
0 & 0 & -2 & -1 \\
-i & 1 & -1 & 1
\end{pmatrix}
\]
are
\[
\xi_1 = -2.376078978, \\
\xi_2 = -0.595187944, \\
\xi_3 = 0.595187944, \\
\xi_4 = 2.376078978,
\]
and the eigenvalues of
\[
\begin{pmatrix}
-1 & -2 \\
-2 & 1
\end{pmatrix}
\]
are
\[
\rho_1 = -2.236067977, \\
\rho_2 = 2.236067977.
\]
Notice that
\[-\infty < \xi_1 < \rho_1 < \xi_2 < \xi_3 < \rho_2 < \xi_4 < +\infty .\]
The intersection of graphics of \(f(x)\) and of \(g(x)\) is:
Finally, if we evaluate directly the eigenvalues of $A$ we get:

$$
\begin{align*}
\theta_1 &= -2.677387677, \\
\theta_2 &= -2.349613205, \\
\theta_3 &= -0.730287931, \\
\theta_4 &= 0.473357313, \\
\theta_5 &= 1.309649492, \\
\theta_6 &= 2.282754449, \\
\theta_7 &= 3.691527560.
\end{align*}
$$

7. Multiplicities of eigenvalues

For any square matrix $A$, let $m_A(\theta)$ denote the (algebraic) multiplicity of $\theta$ as an eigenvalue of $A$. The interlacing established in Theorem 3.5 has a straightforward consequence for $A \in \mathcal{H}(T)$, where $T$ is a tree:

$$
|m_{A(T\setminus i)}(\theta) - m_{A(T)}(\theta)| \leq 1.
$$

Suppose $m_{A(T)}(\theta) > 0$. A vertex $i$ of $T$ is $\theta$-essential and $\theta$-positive if $m_{A(T\setminus i)}(\theta) = m_{A(T)}(\theta) - 1$ and $m_{A(T\setminus i)}(\theta) = m_{A(T)}(\theta) + 1$, respectively. If $\theta$ is understood, we shall omit the mention of $\theta$.

Note that each $A \in \mathcal{H}(T)$ has at least one essential vertex. Indeed, the multiplicity of $\theta$ as zero of $\varphi'(T, x)$ is $m_{A(T)}(\theta) - 1$. If $m_{A(T\setminus i)}(\theta) \geq m_{A(T)}(\theta)$, for all vertices $i$ in $T$, then by Corollary 3.4 the multiplicity of $\theta$ as zero of $\varphi'(T, x)$ is at least $m_{A(T)}(\theta)$.

The next theorem states that if $P$ is a path in a tree $T$, then $\varphi(T\setminus P, x)/\varphi(T, x)$ has only simple poles.

**Theorem 7.1.** Let $P$ be a path in the tree $T$ and let $A \in \mathcal{H}(T)$. If $\theta$ is an eigenvalue of $A(T)$, then

$$
(7.1) \quad m_{A(T\setminus P)}(\theta) \geq m_{A(T)}(\theta) - 1.
$$

**Proof.** Suppose $\theta$ is an eigenvalue of $A(T)$ with $m_A(\theta) > 1$. Then $\theta$ is a zero of $\varphi'(T, x)^2 - \varphi''(T, x)\varphi(T, x)$ with multiplicity at least $2m_A(\theta) - 2$. From (4.3), $\theta$ is a zero of nonnegative summation $\sum_{i,j=1}^n W(P_{ij}) \varphi(T\setminus P_{ij}, x)^2$, and therefore $\theta$ has multiplicity of each $\varphi(T\setminus P_{ij}, x)$ at least $m_A(\theta) - 1$. $\square$

In general, we say that a path $P$ is $P$-essential if $m_{A(T\setminus P)}(\theta) = m_{A(T)}(\theta) - 1$. Given any $\theta$-essential vertex $i$, with $\theta \neq a_{ii}$, there is an adjacent vertex $j$, such that the path $ij$ is essential. For, if $m_{A(T\setminus ik)}(\theta) \geq m_{A(T)}(\theta)$, for all $k \sim i$, then by (3.2), $m_{A(T\setminus P)}(\theta) \geq m_{A(T)}(\theta)$, which is a contradiction.

We also point out that if a path $P_{ij}$ is essential, for some vertex $j$, then $i$ is an essential vertex. In fact, suppose that $m_{A(T\setminus j)}(\theta) \geq m_{A(T)}(\theta)$. For any $j \neq i$, the multiplicity of $\theta$ as a zero of $\varphi(T\setminus i, x)\varphi(T\setminus j, x) - \varphi(T\setminus ij, x)\varphi(T, x)$ is at least $2m_{A(T)}(\theta) - 1$. It is in fact, by (4.4), at least $2m_{A(T)}(\theta)$, and therefore $m_{A(T\setminus P_{ij})}(\theta) \geq m_{A(T)}(\theta)$.

There are some recent results on the multiplicities of eigenvalues of a tree inspired by matching theory (cf. [4]). For example Johnson and Leal Duarte [11] proved that the maximum multiplicity of any single eigenvalue among all matrices in $\mathcal{H}(T)$ is equal to the smallest number of vertex disjoint paths of $T$ that cover all the vertices of $T$. Later, they also proved that the number of distinct eigenvalues of a Hermitian matrix whose graph is the tree $T$ is at least the number of vertices in a longest path in $T$ (cf. [14]).

**Lemma 7.2.** Let $T$ be (weighted) tree and $i$ a non-essential vertex in $T$. Then $i$ is positive if and only if there exists $j \sim i$ essential in $T\setminus i$. 


Proof. Suppose $i$ is $\theta$-positive and $m_{A(T\setminus ij)}(\theta) > m_{A(T)}(\theta)$, for all $j \sim i$. Then (3.2) leads to a contradiction. Conversely, by a remark of the last section, the path $ij$, for any $j$, is not essential and, consequently, $m_{A(T\setminus ij)}(\theta) \geq m_{A(T)}(\theta)$. Suppose that $j \sim i$ is essential in $T\setminus i$. Then $m_{A(T\setminus ij)}(\theta) = m_{A(T\setminus i)}(\theta) - 1$. Hence $i$ is positive. □

Lemma 7.3. Let $i$ and $j$ be two adjacent vertices in a weighted tree $T$ such that $\varphi(T\setminus i, x)$ and $\varphi(T\setminus ij, x)$ have no common zero. Then $\varphi(T, x)$ and $\varphi(T\setminus i, x)$ also have no common zero.

Proof. Suppose $\theta$ is a common zero of $\varphi(T\setminus i, x)$ and $(T\setminus ij, x)$. By Theorem 7.1, if $m_{A(T)}(\theta) > 1$, then $\theta$ is a common zero of $\varphi(T\setminus i, x)$ and $\varphi(T\setminus ij, x)$. If $m_{A(T\setminus i)}(\theta) > 1$, by interlacing we have $m_{A(T\setminus ij)}(\theta) > 0$. Therefore, $m_{A(T)}(\theta) = m_{A(T\setminus i)}(\theta) = 1$. From Lemma 7.2, there is no adjacent vertex to $i$ essential in $T\setminus i$. Thus $m_{A(T\setminus ij)}(\theta) \geq m_{A(T)}(\theta) = 1$, and $\theta$ is a zero of $\varphi(T\setminus ij, x)$ which is a contradiction. □

Under the conditions of the above lemma, by interlacing, we conclude that $\varphi(T, x)$ and $\varphi(T\setminus i, x)$ have only simple zeros. By induction we also have the following corollary:

Corollary 7.4. Let $H$ be an induced subgraph of $T$ and suppose that there is a common vertex, $i$, to $H$ and to a path $P$ in $T$ and the union of the vertices of $H$ and $P$ is $V(T)$. If $\varphi(H, x)$ and $\varphi(H\setminus i, x)$ have no common root, then all eigenvalues of $A(T)$ are simple.

Consider the matrix

$$
\begin{pmatrix}
1 & 0 & i & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
-i & -1 & -2 & 1 & 0 \\
0 & 0 & 1 & 2 & \sqrt{2} \\
0 & 0 & 0 & -\sqrt{2} & 3
\end{pmatrix}
$$

whose graph is the tree

![Fig. 3](image)

The matrix has eigenvalue 1 with multiplicity 2. Therefore

$$
\begin{pmatrix}
1 & 0 & i \\
0 & 1 & -1 \\
-i & -1 & -2
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$

have at least one common eigenvalue.

We observe that not all results on matching polynomials can be extended to characteristic polynomials of general (weighted) graphs or, even, to (weighted) trees as the results that we have seen in this section.

References


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