Operads on noncrossing partitions

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Let X be a random variable. Its n-th moment is the expected value of X^n (if it exists):

$$m_n(X) = \mathbb{E}(X_n).$$

Let X and Y be two independent random variables. Then:

$$m_n(XY) = m_n(X)m_n(Y).$$

This allows to determine the law of $X + Y$ thanks to its moments:

$$m_n(X + Y) = \sum_{k=0}^{n} \binom{n}{k} m_k(X)m_{n-k}(Y).$$

The variance of X is:

$$V(X) = m_2(X) - m_1(X)^2.$$

If X and Y are independent:

$$V(X+Y) = V(X) + V(Y).$$

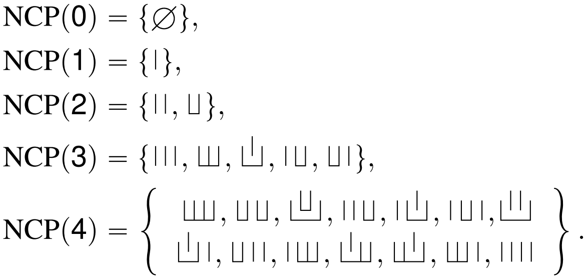
For noncommuting random variables (for example, taking their values in matrices algebras), one uses instead the notion of free independence [Nika, Speicher].
This can be defined using free cumulants associated to a random variable. They are related to the moments through the following formula:

$$m_n = \sum_{Q \in \mathrm{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|},$$

where $\mathrm{NCP}(n)$ is the set of noncrossing partitions of $[n]$.
Under reasonable conditions, X and Y are freely independent if, and only if, for any n,

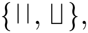
$$k_n(X+Y) = k_n(X) + k_n(Y).$$

We denote by $\mathrm{NCP}(n)$ the set of noncrossing partitions on $[n]$. They are represented by diagrams:



$\mathrm{NCP}(1) =$ 

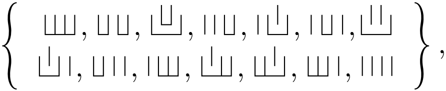
$$m_1 = k_1,$$

$\mathrm{NCP}(2) =$ 

$$m_2 = k_1^2 + k_2,$$

$\mathrm{NCP}(3) =$ 

$$m_3 = k_1^3 + 3k_1k_2 + k_3,$$

$\mathrm{NCP}(4) =$ 

$$m_4 = k_1^4 + 4k_1k_3 + 6k_1^2k_2 + 2k_2^2 + k_4.$$

$$\begin{cases} m_1 &= k_1, \\ m_2 &= k_1^2 + k_2, \\ m_3 &= k_1^3 + 3k_1k_2 + k_3, \\ m_4 &= k_1^4 + 4k_1k_3 + 6k_1^2k_2 + 2k_2^2 + k_4, \end{cases}$$

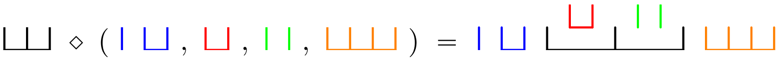
$$\Longleftrightarrow \begin{cases} k_1 &= m_1, \\ k_2 &= m_2 - m_1^2, \\ k_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\ k_4 &= m_4 - 2m_2^2 - 4m_1m_3 + 10m_1^2m_2 - 5m_1^4. \end{cases}$$

$$m_n = \sum_{Q \in \mathrm{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|}.$$

The aim is to "understand" this formula through the formalism of operads and bialgebras.

First operadic structure: insertion of noncrossing partitions in gaps of a noncrossing partition

Any element $p \in \mathrm{NCP}(n)$ has $n+1$ gaps to insert other noncrossing partitions, from left to right: the first one on the left of the noncrossing partition, the last one on the right.



Proposition

With this composition, the sequence $\mathcal{NCP}_0(n) = \mathbb{K}\mathrm{NCP}(n-1)$ is a nonsymmetric operad denoted by $\mathcal{NCP}_0$.

$$\diamond : \bigoplus_{n=1}^{\infty} \mathcal{NCP}_0(n) \otimes \mathcal{NCP}_0^{\otimes n} \longrightarrow \mathcal{NCP}_0.$$

Unit of this operad: $\emptyset \in \mathrm{NCP}(0) \subseteq \mathbb{K}\mathcal{NCP}(1)$.

Reminders on operads

A (set-theoretic, nonsymmetric) operad is a family $(P(n))_{n\in\mathbb{N}}$ of sets, with maps

$$\circ : \begin{cases} P(n) \times P(k_1) \times \ldots \times P(k_n) & \longrightarrow & P(k_1 + \ldots + k_n) \\ (p, p_1, \ldots, p_n) & \longrightarrow & p \circ (p_1, \ldots, p_n) \end{cases}$$

such that:

1. Associativity:

$$\begin{aligned} &(p \circ (p_1, \ldots, p_n)) \circ (p_{1,1}, \ldots p_{1,k_1}, \ldots, p_{n,1}, \ldots, p_{n,k_n}) \\ &= p \circ (p_1 \circ (p_{1,1}, \ldots p_{1,k_1}), \ldots, p_n) \circ (p_{n,1}, \ldots, p_{n,k_n})). \end{aligned}$$

2. Unit: $I \in P(1)$ such that $I \circ p = p \circ (I, \ldots, I) = p$ for any $p \in P(n)$.

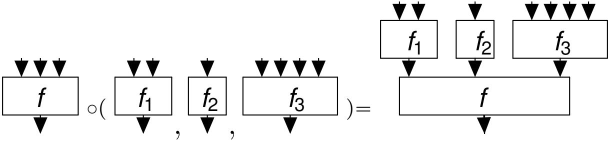
Reminders on operads

Maps

If E is a set, $P_E(n)$ is the set of maps from E^n to E.

$$f \circ (f_1, \ldots, f_n) : E^{k_1+\ldots+k_n} \longrightarrow E.$$

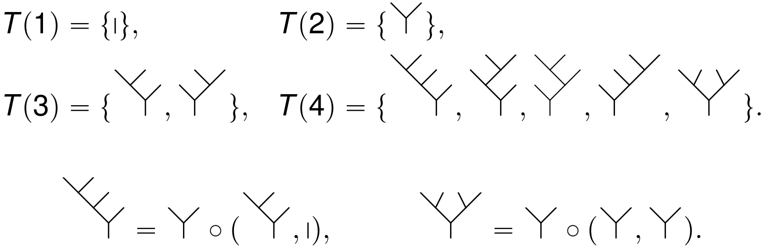
The unit is Id_E.



Reminders on operads

Trees

$T(n)$ is the set of planar binary trees with n leaves. The tree $T \circ (T_1, \ldots, T_n)$ is obtained by grafting T_i on the i-th leaf of T for all i.



Dually, we obtain a coassociative coproduct on noncrossing partitions:

$$\begin{aligned}
\Delta_0(\emptyset) &= \emptyset \otimes \emptyset,\\
\Delta_0(|) &= | \otimes \emptyset\emptyset + \emptyset \otimes |,\\
\Delta_0(\sqcup) &= \sqcup \otimes \emptyset\emptyset\emptyset + \emptyset \otimes \sqcup,\\
\Delta_0(||) &= || \otimes \emptyset\emptyset\emptyset + | \otimes (|\emptyset + \emptyset|) + \emptyset \otimes ||,\\
\Delta_0(\sqcup\!\!\!|) &= \sqcup\!\!\!| \otimes \emptyset\emptyset\emptyset\emptyset + \sqcup \otimes \emptyset|\emptyset + \emptyset \otimes \sqcup\!\!\!|.
\end{aligned}$$

$T(\mathcal{NCP}_0)$ is a bialgebra. It is graded by the number of blocks. Its counit is given by

$$\forall P \in \mathrm{NCP}_0, \qquad \varepsilon(P) = \delta_{P,\emptyset}.$$

Let us identify $\emptyset$ and the unit of $T(\mathcal{NCP}_0)$:

$$
\begin{aligned}
\Delta(\mid) &= \mid \otimes 1 + 1 \otimes \mid,\\
\Delta(\sqcup) &= \sqcup \otimes 1 + 1 \otimes \sqcup,\\
\Delta(\mid\mid) &= \mid\mid \otimes 1 + 2 \mid \otimes \mid + 1 \otimes \mid\mid,\\
\Delta(\overset{|}{\sqcup}) &= \overset{|}{\sqcup} \otimes 1 + \sqcup \otimes \mid + 1 \otimes \overset{|}{\sqcup}.
\end{aligned}
$$

$T(\mathcal{NCP})$ is a graded bialgebra:

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta.$$

Its counit is given by

$$\forall P \in \mathrm{NCP}, \qquad \varepsilon(P) = 0.$$

More structure for almost free:

$$\Delta_{\prec}(P) = \sum_{\substack{(L,U)\in \mathrm{cut}(P),\\ 1\in L}} L \otimes \dot{U},$$
$$\Delta_{\succ}(P) = \sum_{\substack{(L,U)\in \mathrm{cut}(P),\\ 1\in U}} L \otimes \dot{U}.$$

Then:

$$(\Delta \otimes Id) \circ \Delta_{\succ} = (Id \otimes \Delta_{\succ}) \circ \Delta_{\succ},$$
$$(\Delta_{\succ} \otimes Id) \circ \Delta_{\prec} = (Id \otimes \Delta_{\prec}) \otimes \Delta_{\succ},$$
$$(\Delta_{\prec} \otimes Id) \circ \Delta_{\prec} = (Id \otimes \Delta) \circ \Delta_{\prec}.$$

This is a unshuffle bialgebra (or codendriform bialgebra).

As a consequence, the dual algebra $T(\mathcal{NCP})^*$ is a dendriform algebra, with convolution product $\star = \prec + \succ$.

An example

Let κ be the infinitesimal character on $T(\mathcal{NCP})$ such that for any noncrossing partition P,

$$\kappa(P) = \begin{cases} k_n \text{ if } P \text{ has only one block of size } n, \\ 0 \text{ otherwise.} \end{cases}$$

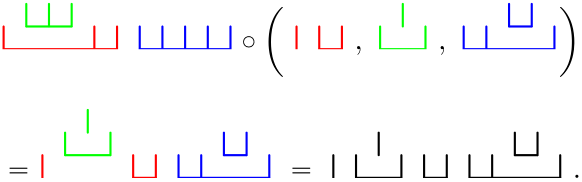
There exists a unique $\phi \in T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa \prec \phi.$$

Then ϕ is a character, called the left half-exponential of κ. For any non crossing partition Q:

Second operadic structure: block substitution

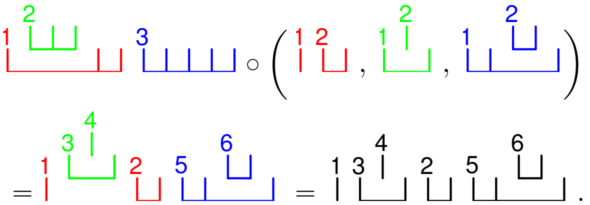
We want to replace any block of a noncrossing partition by another noncrossing partition:



First technical difficulty: how to know which block is replaced by each noncrossing partition?

Second operadic structure: block substitution

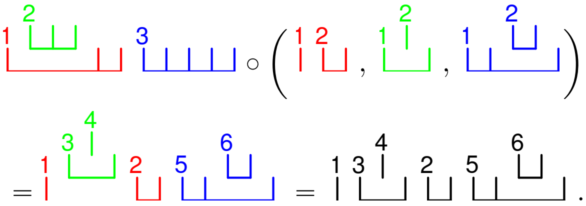
We want to replace any block of a noncrossing composition by another noncrossing compositions:



The numbers indicate the index of the blocks in the noncrossing partitions.

Second operadic structure: block substitution

We want to replace any block of a noncrossing composition by another noncrossing compositions:



Second technical difficulty: some compositions are not possible. Each block has a color (its size); we can only substitute to a block of color n a noncrossing composition of degree n.

Let us denote by $\mathrm{NCC}(n; k_1, \ldots, k_p)$ the set of noncrossing compositions on $[n]$ with p blocks, the i-th block of size k_i. The composition is defined from

$$\mathrm{NCC}(n; k_1, \ldots, k_p) \otimes \mathrm{NCC}(k_1; \underline{\ell}_1) \otimes \ldots \otimes \mathrm{NCC}(k_p; \underline{\ell}_p)$$

to

$$\mathrm{NCC}(k_1 + \ldots + k_p; \underline{\ell}_1, \ldots, \underline{\ell}_p).$$

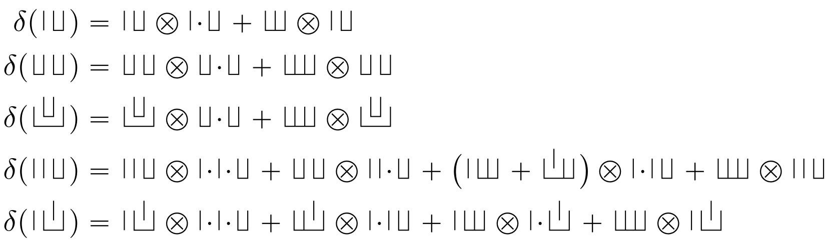
Proposition

With this composition, the sequence $\mathcal{NCC}(n; k_1, \ldots, k_p) = \mathbb{K}\mathrm{NCC}(n; k_1, \ldots, k_p)$ is a colored operad.

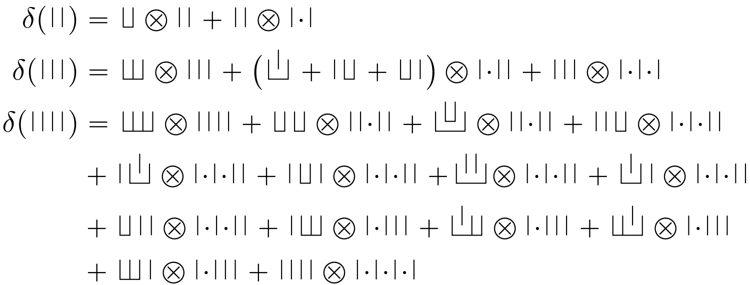
Partial unit $\in \mathcal{NCC}(n; n)$: $I_n = ([n])$.

$$I_1 = |, \qquad I_2 = \sqcup, \qquad I_3 = \sqcup\!\sqcup, \qquad I_4 = \sqcup\!\sqcup\!\sqcup \ldots$$

Dually (with technical difficulties), we obtain a second coproduct on $S(\mathcal{NCP})$:

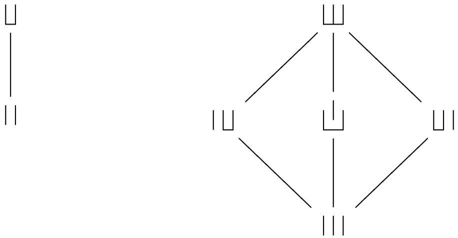


The finest partitions have the longest formulas for the coproduct:



Partial order on noncrossing partitions

Let P, Q be two noncrossing partitions. Then $P \leqslant Q$ if P is a refinement of Q.



Working similarly with noncrossing compositions,

$$P \leqslant Q \Longleftrightarrow \exists P_1, \ldots, P_n,\ P = Q \circ (P_1, \ldots, P_n).$$

$(P_1, \ldots, P_n)$ is unique up to their order, and we put in $S(\mathcal{NCP})$:

$$P/Q = P_1 \ldots P_n.$$

Then, in $S(\mathcal{NCP})$:

$$\delta(P) = \sum_{Q \geqslant P} Q \otimes P/Q.$$

We recover an incidence coalgebra in the sense of Schmitt.

The coproduct δ induces a second convolution $*$ product on $S(\mathcal{NCP})^*$.

$$m_n = \sum_{Q \geqslant J_n} k_{J_n/Q} = \zeta * k(J_n),$$

where $\zeta(P) = 1$ for any noncrossing partition and

$$J_n = \{\{1\}, \ldots, \{n\}\} = \underbrace{| \ldots |}_{n}.$$

$$J_1 = |, \qquad J_2 = ||, \qquad J_3 = |||, \qquad J_4 = ||||, \ldots$$

If $\alpha, \beta \in S(\mathcal{NCP})^*$ and γ is a character of $S(\mathcal{NCP})$:

$$(\alpha \star \beta) * \gamma = (\alpha * \gamma) \star (\beta * \gamma),$$
$$(\alpha \prec \beta) * \gamma = (\alpha * \gamma) \prec (\beta * \gamma),$$
$$(\alpha \succ \beta) * \gamma = (\alpha * \gamma) \succ (\beta * \gamma).$$

For any infinitesimal character κ, let us denote by $\mathcal{E}_{\prec}(\kappa)$ the unique element ϕ of $T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa \prec \phi.$$

From the dendriform axioms, ϕ is a character of $T(\mathcal{NCP})$ or of $S(\mathcal{NCP})$.

The map $\mathcal{E}_{\prec}$ is the left half-exponential.

An application

Let e be the infinitesimal character on $S(\mathcal{NCP})$ defined by:

$$e(P) = \begin{cases} 1 \text{ if } P \text{ has one block,} \\ 0 \text{ otherwise.} \end{cases}$$

Then π_1 is the block of P containing 1,

$$\mathcal{E}_{\prec}(e)(P) = 1 \times \mathcal{E}_{\prec}(e)(P \backslash \{\pi_1\}).$$

Hence, for any noncrossing partition P,

$$\mathcal{E}_{\prec}(e)(P) = 1 = \zeta(P).$$

An application

Proposition

Let κ be an infinitesimal character and ψ be a character of $S(\mathcal{NCP})$. We denote by K the character such that for any noncrossing partition P, $K(P) = \kappa(P)$. Then:

$$\phi = \mathcal{E}_{\prec}(\kappa) \Longleftrightarrow \phi = \zeta * K.$$

K is the unique character such that $e \prec K = \kappa$.

An application

Let $\psi = \zeta * K$. Then:

$$\begin{aligned} \psi &= \mathcal{E}_{\prec}(e) * K \\ &= (\varepsilon + e \prec \zeta) * K \\ &= \varepsilon * K + (e \prec \zeta) * K \\ &= \varepsilon + (e * K) \prec (\zeta * K) \\ &= \varepsilon + \kappa \prec \psi. \end{aligned}$$

So $\psi = \mathcal{E}_{\prec}(\kappa)$.

An application

Consequently, if $\psi = \mathcal{E}_{\prec}(\kappa)$:

$$\psi(J_n) = \sum_{Q \in \mathrm{NCP}(n)} \prod_{\pi \in Q} k(|J_{|\pi|}).$$

To be compared to

$$m_n = \sum_{Q \in \mathrm{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|}.$$

Corollary

The character giving moments is the left half-exponential of the infinitesimal character giving free cumulants.

Reference

Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, Frédéric Patras:

Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations

`https://arxiv.org/abs/1907.01190`

To be published in Advances in Maths.

Thank you for your attention!