

Operads on noncrossing partitions

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Let X be a random variable. Its n -th moment is the expected value of X^n (if it exists):

$$m_n(X) = \mathbb{E}(X^n).$$

Let X and Y be two independent random variables. Then:

$$m_n(XY) = m_n(X)m_n(Y).$$

This allows to determine the law of $X + Y$ thanks to its moments:

$$m_n(X + Y) = \sum_{k=0}^n \binom{n}{k} m_k(X) m_{n-k}(Y).$$

The variance of X is:

$$V(X) = m_2(X) - m_1(X)^2.$$

If X and Y are independent:

$$V(X + Y) = V(X) + V(Y).$$

For noncommuting random variables (for example, taking their values in matrices algebras), one uses instead the notion of free independence [Nika, Speicher].

This can be defined using free cumulants associated to a random variable. They are related to the moments through the following formula:

$$m_n = \sum_{Q \in \text{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|},$$

where $\text{NCP}(n)$ is the set of noncrossing partitions of $[n]$.

Under reasonable conditions, X and Y are freely independent if, and only if, for any n ,

$$k_n(X + Y) = k_n(X) + k_n(Y).$$

We denote by $\text{NCP}(n)$ the set of noncrossing partitions on $[n]$.
They are represented by diagrams:

$$\text{NCP}(0) = \{\emptyset\},$$

$$\text{NCP}(1) = \{|\},$$

$$\text{NCP}(2) = \{||, \cup\},$$

$$\text{NCP}(3) = \{|||, \cup\cup, \cup|\},$$

$$\text{NCP}(4) = \left\{ \begin{array}{l} \cup\cup\cup, \cup\cup\cup, \cup\cup|\}, \\ \cup|\cup, \cup|\cup, \cup|\cup \end{array} \right\}.$$

$$\text{NCP}(1) = \{|\},$$

$$m_1 = k_1,$$

$$\text{NCP}(2) = \{||, \sqcup\},$$

$$m_2 = k_1^2 + k_2,$$

$$\text{NCP}(3) = \{|||, \sqcup\sqcup, \sqcup|\}, \{|\sqcup, \sqcup|\},$$

$$m_3 = k_1^3 + 3k_1k_2 + k_3,$$

$$\text{NCP}(4) = \left\{ \begin{array}{l} \sqcup\sqcup\sqcup, \sqcup\sqcup\sqcup, \sqcup\sqcup|\}, \{|\sqcup\sqcup, |\sqcup|\}, \{|\sqcup|\sqcup, \sqcup|\sqcup\} \\ \sqcup|\sqcup, \sqcup|\sqcup, |\sqcup\sqcup, \sqcup|\sqcup, \sqcup|\sqcup, \sqcup|\sqcup, \sqcup|\sqcup\} \end{array} \right\},$$

$$m_4 = k_1^4 + 4k_1k_3 + 6k_1^2k_2 + 2k_2^2 + k_4.$$

$$\begin{cases} m_1 = k_1, \\ m_2 = k_1^2 + k_2, \\ m_3 = k_1^3 + 3k_1k_2 + k_3, \\ m_4 = k_1^4 + 4k_1k_3 + 6k_1^2k_2 + 2k_2^2 + k_4, \end{cases}$$

$$\iff \begin{cases} k_1 = m_1, \\ k_2 = m_2 - m_1^2, \\ k_3 = m_3 - 3m_1m_2 + 2m_1^3, \\ k_4 = m_4 - 2m_2^2 - 4m_1m_3 + 10m_1^2m_2 - 5m_1^4. \end{cases}$$

$$m_n = \sum_{Q \in \text{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|}.$$

The aim is to "understand" this formula through the formalism of operads and bialgebras.

First operadic structure: insertion of noncrossing partitions in gaps of a noncrossing partition

Any element $p \in \text{NCP}(n)$ has $n + 1$ gaps to insert other noncrossing partitions, from left to right: the first one on the left of the noncrossing partition, the last one on the right.

$$\text{Diagram illustrating the gap insertion operation. On the left, a black noncrossing partition with two arcs is shown. This is followed by a diamond symbol and a tuple of four noncrossing partitions: a blue one with one arc, a red one with one arc, a green one with two arcs, and an orange one with three arcs. An equals sign follows, leading to the result: the original black partition with the four colored partitions inserted into its four gaps from left to right. The blue partition is in the first gap, the red one in the second, the green one in the third, and the orange one in the fourth.$$

Proposition

With this composition, the sequence $\mathcal{NCP}_0(n) = \mathbb{K}\mathcal{NCP}(n-1)$ is a nonsymmetric operad denoted by \mathcal{NCP}_0 .

$$\diamond : \bigoplus_{n=1}^{\infty} \mathcal{NCP}_0(n) \otimes \mathcal{NCP}_0^{\otimes n} \longrightarrow \mathcal{NCP}_0.$$

Unit of this operad: $\emptyset \in \mathcal{NCP}(0) \subseteq \mathbb{K}\mathcal{NCP}(1)$.

Reminders on operads

A (set-theoretic, nonsymmetric) operad is a family $(P(n))_{n \in \mathbb{N}}$ of sets, with maps

$$\circ : \begin{cases} P(n) \times P(k_1) \times \dots \times P(k_n) & \longrightarrow P(k_1 + \dots + k_n) \\ (\rho, \rho_1, \dots, \rho_n) & \longrightarrow \rho \circ (\rho_1, \dots, \rho_n) \end{cases}$$

such that:

① **Associativity:**

$$\begin{aligned} & (\rho \circ (\rho_1, \dots, \rho_n)) \circ (\rho_{1,1}, \dots, \rho_{1,k_1}, \dots, \rho_{n,1}, \dots, \rho_{n,k_n}) \\ &= \rho \circ (\rho_1 \circ (\rho_{1,1}, \dots, \rho_{1,k_1}), \dots, \rho_n \circ (\rho_{n,1}, \dots, \rho_{n,k_n})). \end{aligned}$$

② **Unit:** $I \in P(1)$ such that $I \circ \rho = \rho \circ (I, \dots, I) = \rho$ for any $\rho \in P(n)$.

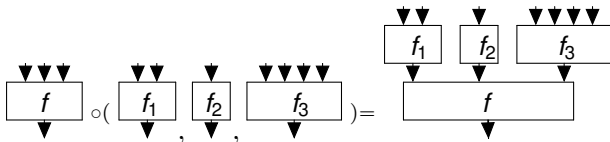
Reminders on operads

Maps

If E is a set, $P_E(n)$ is the set of maps from E^n to E .

$$f \circ (f_1, \dots, f_n) : E^{k_1 + \dots + k_n} \longrightarrow E.$$

The unit is Id_E .



Reminders on operads

Trees

$T(n)$ is the set of planar binary trees with n leaves. The tree $T \circ (T_1, \dots, T_n)$ is obtained by grafting T_i on the i -th leaf of T for all i .

$$T(1) = \{|\},$$

$$T(2) = \{ \begin{array}{c} \diagup \diagdown \\ | \end{array} \},$$

$$T(3) = \{ \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array} \},$$

$$T(4) = \{ \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array} \}.$$

$$\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \diagup \diagdown \\ | \end{array} \circ (\begin{array}{c} \diagup \diagdown \\ | \end{array}, |\},$$

$$\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array} = \begin{array}{c} \diagup \diagdown \\ | \end{array} \circ (\begin{array}{c} \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ | \end{array}).$$

Dually, we obtain a coassociative coproduct on noncrossing partitions:

$$\Delta_0(\emptyset) = \emptyset \otimes \emptyset,$$

$$\Delta_0(|) = | \otimes \emptyset \emptyset + \emptyset \otimes |,$$

$$\Delta_0(\sqcup) = \sqcup \otimes \emptyset \emptyset \emptyset + \emptyset \otimes \sqcup,$$

$$\Delta_0(||) = || \otimes \emptyset \emptyset \emptyset + | \otimes (| \emptyset + \emptyset |) + \emptyset \otimes ||,$$

$$\Delta_0(\sqcup\sqcup) = \sqcup\sqcup \otimes \emptyset \emptyset \emptyset \emptyset + \sqcup \otimes \emptyset | \emptyset + \emptyset \otimes \sqcup\sqcup.$$

$T(\mathcal{NCP}_0)$ is a bialgebra. It is graded by the number of blocks. Its counit is given by

$$\forall P \in \mathcal{NCP}_0,$$

$$\varepsilon(P) = \delta_{P, \emptyset}.$$

Let us identify \emptyset and the unit of $T(\mathcal{NCP}_0)$:

$$\Delta(|) = | \otimes \mathbf{1} + \mathbf{1} \otimes |,$$

$$\Delta(\sqcup) = \sqcup \otimes \mathbf{1} + \mathbf{1} \otimes \sqcup,$$

$$\Delta(\parallel) = \parallel \otimes \mathbf{1} + 2| \otimes | + \mathbf{1} \otimes \parallel,$$

$$\Delta(\sqcup\!\!\!\downarrow) = \sqcup\!\!\!\downarrow \otimes \mathbf{1} + \sqcup \otimes | + \mathbf{1} \otimes \sqcup\!\!\!\downarrow.$$

$T(\mathcal{NCP})$ is a graded bialgebra:

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta.$$

Its counit is given by

$$\forall P \in \mathcal{NCP}, \quad \varepsilon(P) = 0.$$

More structure for almost free:

$$\Delta_{<}(P) = \sum_{\substack{(L,U) \in \text{cut}(P), \\ 1 \in L}} L \otimes \dot{U},$$

$$\Delta_{>}(P) = \sum_{\substack{(L,U) \in \text{cut}(P), \\ 1 \in U}} L \otimes \dot{U}.$$

Then:

$$\begin{aligned} (\Delta \otimes \text{Id}) \circ \Delta_{>} &= (\text{Id} \otimes \Delta_{>}) \circ \Delta_{>}, \\ (\Delta_{>} \otimes \text{Id}) \circ \Delta_{<} &= (\text{Id} \otimes \Delta_{<}) \otimes \Delta_{>}, \\ (\Delta_{<} \otimes \text{Id}) \circ \Delta_{<} &= (\text{Id} \otimes \Delta) \circ \Delta_{<}. \end{aligned}$$

This is a unshuffle bialgebra (or codendriform bialgebra).

As a consequence, the dual algebra $T(\mathcal{NCP})^*$ is a dendriform algebra, with convolution product $\star = \langle + \rangle$.

An example

Let κ be the infinitesimal character on $T(\mathcal{NCP})$ such that for any noncrossing partition P ,

$$\kappa(P) = \begin{cases} k_n & \text{if } P \text{ has only one block of size } n, \\ 0 & \text{otherwise.} \end{cases}$$

There exists a unique $\phi \in T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa \langle \phi.$$

Then ϕ is a character, called the left half-exponential of κ . For any non crossing partition Q :

Second operadic structure: block substitution

We want to replace any block of a noncrossing partition by another noncrossing partition:

$$\begin{aligned}
 & \left(\text{red block} \right) \circ \left(\text{blue block} \right) \circ \left(\text{green block} \right) \\
 &= \text{red block} \circ \left(\text{green block} \right) \circ \text{blue block} = \text{red block} \circ \left(\text{black block} \right)
 \end{aligned}$$

The diagram shows the substitution of a block in a noncrossing partition. The first row shows a red block with two arcs, a blue block with four arcs, and a green block with one arc. These are combined via a composition operation (indicated by a circle with a dot) to form a new partition where the red block is replaced by the green block. The second row shows the resulting partition as a red block with one arc, a green block with one arc, and a blue block with four arcs, which is equivalent to a single black partition with one arc, one arc, and four arcs.

First technical difficulty: how to know which block is replaced by each noncrossing partition?

Second operadic structure: block substitution

We want to replace any block of a noncrossing composition by another noncrossing compositions:

$$\begin{array}{c}
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \quad
 \begin{array}{c} 3 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 2 \end{array}
 \quad
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \quad
 \circ \quad
 \left(
 \begin{array}{c} 1 \quad 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 2 \end{array}
 ,
 \begin{array}{c} 1 \quad 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 2 \end{array}
 ,
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \right) \\
 \\
 = \begin{array}{c} 4 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \begin{array}{c} 6 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 2 \end{array}
 = \begin{array}{c} 4 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 3 \end{array}
 \begin{array}{c} 6 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \quad 2 \end{array}
 .
 \end{array}$$

The numbers indicate the index of the blocks in the noncrossing partitions.

Second operadic structure: block substitution

We want to replace any block of a noncrossing composition by another noncrossing compositions:

$$\begin{array}{c}
 \begin{array}{c} 2 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \end{array} \quad \begin{array}{c} 3 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \end{array} \circ \left(\begin{array}{c} 1 \ 2 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array}, \begin{array}{c} 2 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array}, \begin{array}{c} 2 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array} \right) \\
 \\
 = \begin{array}{c} 4 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \end{array} \begin{array}{c} 3 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array} \begin{array}{c} 2 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array} \begin{array}{c} 6 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \end{array} = \begin{array}{c} 1 \ 3 \ 4 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array} \begin{array}{c} 2 \\ \text{┌──┐} \\ \text{└──┘} \\ 1 \end{array} \begin{array}{c} 5 \ 6 \\ \text{┌───┐} \\ \text{└───┘} \\ 1 \end{array}.
 \end{array}$$

Second technical difficulty: some compositions are not possible. Each block has a color (its size); we can only substitute to a block of color n a noncrossing composition of degree n .

Let us denote by $\text{NCC}(n; k_1, \dots, k_p)$ the set of noncrossing compositions on $[n]$ with p blocks, the i -th block of size k_i . The composition is defined from

$$\text{NCC}(n; k_1, \dots, k_p) \otimes \text{NCC}(k_1; \underline{\ell}_1) \otimes \dots \otimes \text{NCC}(k_p; \underline{\ell}_p)$$

to

$$\text{NCC}(k_1 + \dots + k_p; \underline{\ell}_1, \dots, \underline{\ell}_p).$$

Proposition

With this composition, the sequence

$\mathcal{NCC}(n; k_1, \dots, k_p) = \mathbb{K}\mathcal{NCC}(n; k_1, \dots, k_p)$ is a colored operad.

Partial unit $\in \mathcal{NCC}(n; n)$: $I_n = ([n])$.

$$I_1 = |, \quad I_2 = \sqcup, \quad I_3 = \sqcup\sqcup, \quad I_4 = \sqcup\sqcup\sqcup \dots$$

Dually (with technical difficulties), we obtain a second coproduct on $\mathcal{S}(\mathcal{NCP})$:

$$\delta(1 \cup) = 1 \cup \otimes 1 \cdot \cup + \cup \otimes 1 \cup$$

$$\delta(\cup \cup) = \cup \cup \otimes \cup \cdot \cup + \cup \cup \otimes \cup \cup$$

$$\delta(\cup \cup) = \cup \cup \otimes \cup \cdot \cup + \cup \cup \otimes \cup \cup$$

$$\delta(1 \cup \cup) = 1 \cup \cup \otimes 1 \cdot 1 \cdot \cup + \cup \cup \otimes 1 \cdot \cup + (1 \cup + \cup \cup) \otimes 1 \cdot 1 \cup + \cup \cup \otimes 1 \cup \cup$$

$$\delta(1 \cup) = 1 \cup \otimes 1 \cdot 1 \cdot \cup + \cup \cup \otimes 1 \cdot 1 \cup + 1 \cup \otimes 1 \cdot \cup + \cup \cup \otimes 1 \cup$$

The finest partitions have the longest formulas for the coproduct:

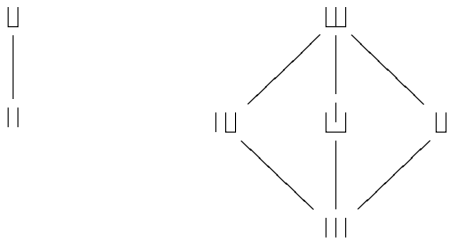
$$\delta(11) = \cup \otimes 11 + 11 \otimes | \cdot |$$

$$\delta(111) = \sqcup \otimes 111 + (\downarrow + | \cup + \cup |) \otimes | \cdot 11 + 111 \otimes | \cdot | \cdot |$$

$$\begin{aligned} \delta(1111) = & \sqcup \otimes 1111 + \cup \cup \otimes 11 \cdot 11 + \downarrow \cup \otimes 11 \cdot 11 + 11 \cup \otimes | \cdot | \cdot 11 \\ & + | \downarrow \otimes | \cdot | \cdot 11 + | \cup | \otimes | \cdot | \cdot 11 + \downarrow \cup \otimes | \cdot | \cdot 11 + \downarrow | \otimes | \cdot | \cdot 11 \\ & + \cup 11 \otimes | \cdot | \cdot 11 + | \sqcup \otimes | \cdot 111 + \downarrow \cup \otimes | \cdot 111 + \downarrow | \otimes | \cdot 111 \\ & + \sqcup | \otimes | \cdot 111 + 1111 \otimes | \cdot | \cdot | \cdot | \end{aligned}$$

Partial order on noncrossing partitions

Let P, Q be two noncrossing partitions. Then $P \leq Q$ if P is a refinement of Q .



Working similarly with noncrossing compositions,

$$P \leq Q \iff \exists P_1, \dots, P_n, P = Q \circ (P_1, \dots, P_n).$$

(P_1, \dots, P_n) is unique up to their order, and we put in $S(\mathcal{NCP})$:

$$P/Q = P_1 \dots P_n.$$

Then, in $S(\mathcal{NCP})$:

$$\delta(P) = \sum_{Q \geq P} Q \otimes P/Q.$$

We recover an incidence coalgebra in the sense of Schmitt.

The coproduct δ induces a second convolution $*$ product on $S(\mathcal{NCP})^*$.

$$m_n = \sum_{Q \geq J_n} k_{J_n/Q} = \zeta * k(J_n),$$

where $\zeta(P) = 1$ for any noncrossing partition and

$$J_n = \{\{1\}, \dots, \{n\}\} = \underbrace{|\dots|}_n.$$

$$J_1 = |, \quad J_2 = ||, \quad J_3 = |||, \quad J_4 = ||||, \dots$$

If $\alpha, \beta \in \mathcal{S}(\mathcal{NCP})^*$ and γ is a character of $\mathcal{S}(\mathcal{NCP})$:

$$(\alpha \star \beta) * \gamma = (\alpha * \gamma) \star (\beta * \gamma),$$

$$(\alpha < \beta) * \gamma = (\alpha * \gamma) < (\beta * \gamma),$$

$$(\alpha > \beta) * \gamma = (\alpha * \gamma) > (\beta * \gamma).$$

For any infinitesimal character κ , let us denote by $\mathcal{E}_{<}(\kappa)$ the unique element ϕ of $T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa < \phi.$$

From the dendriform axioms, ϕ is a character of $T(\mathcal{NCP})$ or of $S(\mathcal{NCP})$.

The map $\mathcal{E}_{<}$ is the left half-exponential.

An application

Let e be the infinitesimal character on $S(\mathcal{NCP})$ defined by:

$$e(P) = \begin{cases} 1 & \text{if } P \text{ has one block,} \\ 0 & \text{otherwise.} \end{cases}$$

Then π_1 is the block of P containing 1,

$$\mathcal{E}_{<}(e)(P) = 1 \times \mathcal{E}_{<}(e)(P \setminus \{\pi_1\}).$$

Hence, for any noncrossing partition P ,

$$\mathcal{E}_{<}(e)(P) = 1 = \zeta(P).$$

An application

Proposition

Let κ be an infinitesimal character and ψ be a character of $\mathcal{S}(\mathcal{NCP})$. We denote by K the character such that for any noncrossing partition P , $K(P) = \kappa(P)$. Then:

$$\phi = \mathcal{E}_{<}(\kappa) \iff \phi = \zeta * K.$$

K is the unique character such that $e < K = \kappa$.

An application

Let $\psi = \zeta * K$. Then:

$$\begin{aligned}
 \psi &= \mathcal{E}_{<}(\mathbf{e}) * K \\
 &= (\varepsilon + \mathbf{e} < \zeta) * K \\
 &= \varepsilon * K + (\mathbf{e} < \zeta) * K \\
 &= \varepsilon + (\mathbf{e} * K) < (\zeta * K) \\
 &= \varepsilon + \kappa < \psi.
 \end{aligned}$$

So $\psi = \mathcal{E}_{<}(\kappa)$.

An application

Consequently, if $\psi = \mathcal{E}_<(\kappa)$:

$$\psi(\mathbf{J}_n) = \sum_{Q \in \text{NCP}(n)} \prod_{\pi \in Q} k(|\mathbf{J}_{|\pi|}).$$

To be compared to

$$m_n = \sum_{Q \in \text{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|}.$$

Corollary

The character giving moments is the left half-exponential of the infinitesimal character giving free cumulants.

Reference

Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, Frédéric Patras:

Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations

<https://arxiv.org/abs/1907.01190>

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Thank you for your attention!