Classes of $(0,1)$-matrices where the Bruhat order and the Secondary Bruhat order coincide

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Let S_n be the symmetric group of degree n, and let $\sigma \in S_n$. We can represent σ as a word by

$$\sigma = \sigma_1 \ldots \sigma_n,$$

with $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$.

A pair (i, j), $1 \le i, j \le n$, is called an inversion of σ if

$$i < j \quad \text{and} \quad \sigma_i > \sigma_j.$$

We can define a partial order $\preceq_B$ on S_n, the Bruhat order, saying

$$\sigma \preceq_B \tau,$$

if σ can be obtained from τ by a sequence of transformations where

$$\tau_1 \dots \tau_i \dots \tau_j \dots \tau_n$$

is replaced by

$$\tau_1 \dots \tau_j \dots \tau_i \dots \tau_n,$$

being (i, j) an inversion of τ.

Let P and Q be two permutation matrices of order n corresponding to permutations π and τ respectively. We write

$$P \preceq_B Q \text{ whenever } \pi \preceq_B \tau.$$

Note that a inversion in a permutation τ corresponds to a submatrix

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the correspondent permutation matrix, Q.

Remove this inversion in τ is equivalent to replace the submatrix L_2 of Q by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, given two permutation matrices P and Q, we say that

$$P \preceq_B Q$$

if P can be obtained from Q by a sequence of one sided interchanges

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There is another way to define the Bruhat order on permutation matrices.

For any m-by-n matrix $A = [a_{i,j}]$ of real entries, let Σ_A denote the m-by-n matrix whose (r, s)-entry is

$$\sigma_{r,s}(A) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{ij}, \quad 1 \leq r \leq m,\ 1 \leq s \leq n.$$

For permutation matrices P and Q of order n we have

$$P \preceq_B Q,$$

if and only if, by the entrywise order

$$\Sigma_P \geq \Sigma_Q.$$

Let $R = (r_1, \ldots, r_m)$, and $S = (s_1, \ldots, s_n)$ be two positive integral vectors such that

$$r_1 + \ldots + r_m = s_1 + \ldots + s_n.$$

We denote by $A(R, S)$ the class of all m-by-n matrices of zeros and ones, the $(0, 1)$-matrices, with row sum vector R, and column sum vector S.

The class of all n-by-n $(0,1)$-matrices with common row and column sum k is denoted by $A(n,k)$.

The class of n-by-n permutation matrices is the class $A(n,1)$.

In 2004, Brualdi and Hwang extended the Bruhat order from $\mathcal{A}(n, 1)$ to any nonempty classes $\mathcal{A}(R, S)$.

Given $A_1, A_2 \in \mathcal{A}(R, S)$ we say that A_1 precedes A_2 by the Bruhat order, and write

$$A_1 \preceq_B A_2$$

if, by the entrywise order,

$$\Sigma_{A_1} \geq \Sigma_{A_2}.$$

Given $A_1, A_2 \in \mathcal{A}(R,S)$, we say that A_1 precedes A_2 by the Secondary Bruhat order, and write

$$A_1 \preceq_{\widehat{B}} A_2,$$

if A_1 can be obtained from A_2 by a sequence of one sided interchanges $L_2 \to I_2$ where

$$L_2 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right],$$

and I_2 is the identity matrix of order 2.

It is straightforward to verify that if $A_1 \preceq_{\widehat{B}} A_2$, then $A_1 \preceq_B A_2$.

In general, the Bruhat order and the Secondary Bruhat order do not coincide on $\mathcal{A}(R, S)$.

They coincide on $\mathcal{A}(n, 1)$, and Brualdi and Deaett proved, that they coincide on $\mathcal{A}(n, 2)$, but, with an example, they showed that the two partial orders are distinct on $\mathcal{A}(6, 3)$.

Theorem

Let $R = (2, 2, \dots, 2)$*, and* $S = (s_1, \dots, s_n)$ *be two nonincreasing positive integral vectors such that* $\mathcal{A}(R, S) \neq \emptyset$*. Then the Bruhat order and the Secondary Bruhat order coincide on* $\mathcal{A}(R, S)$*.*

Lemma(Brualdi and Deaett, 2007)

Let A *and* C *be matrices in* $\mathcal{A}(R,S)$ *with* $A \prec_B C$, *and let* i *and* j *be integers with* $\sigma_{i,j}(A) > \sigma_{i,j}(C)$. *Let* s *and* t *be integers with* (s,t) *lexicographically maximal such that*

$$(r,c) \in \{i,\dots,s-1\} \times \{j,\dots,t-1\} \Longrightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

Then, there exists $(i_0,j_0) \in \{i+1,\dots,s\} \times \{j+1,\dots,t\}$ *with* $a_{i_0 j_0} = 1$.

Lemma(Brualdi, Fernandes and Furtado, 2019)

Let A *and* C *be matrices in* $\mathcal{A}(R,S)$ *with* $A \prec_B C$, *and let* i *and* j *be integers with* $\sigma_{ij}(A) > \sigma_{ij}(C)$. *Let* s *and* t *be integers with* (s,t) *lexicographically minimal such that*

$$(r,c) \in \{s+1,\dots,i\} \times \{t+1,\dots,j\} \Longrightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

Then, there exists $(i_0,j_0) \in \{s+1,\dots,i\} \times \{t+1,\dots,j\}$ *with* $a_{i_0 j_0} = 1$.

Proposition

Let $A,\ C \in \mathcal{A}(R,S)$ *such that* C *covers* A *in the Bruhat order. Let* $p,\ f,\ g$ *and* l *be integers, with* $1 \leq p < l \leq m$*, and* $1 \leq f < g \leq n$*, such that*

$$A[\{p,l\};\{f,g\}] = I_2,$$

and, for any $(r,c) \in \{p,\dots,l-1\} \times \{f,\dots,g-1\}$,

$$\sigma_{rc}(A) > \sigma_{rc}(C).$$

Then $A \preceq_{\widehat{B}} C$.

Outline of the Proof of Theorem 1:

Let A and C be matrices in $\mathcal{A}(R, S)$. We know that if

$$A \prec_{\widehat{B}} C,$$

then

$$A \prec_{B} C.$$

So we need to prove that if $A \prec_{B} C$, then $A \prec_{\widehat{B}} C$.

It suffices to show this when C covers A. So, from now on we assume that C covers A by the Bruhat order.

The strategy is to find integers $p,\ f,\ g$ and l, with $1 \leq p < l \leq m$, and $1 \leq f < g \leq n$, such that

$$A[\{p, l\}; \{f, g\}] = I_2,$$

and for any $(r, c) \in \{p, \dots, l-1\} \times \{f, \dots, g-1\}$,

$$\sigma_{rc}(A) > \sigma_{rc}(C).$$

If this happens, then using Proposition 1 we get $A \prec_{\widehat{B}} C$.

Since $A \prec_B C$, there is a position (i, j) such that $a_{ij} = 1$ and $\sigma_{ij}(A) > \sigma_{ij}(C)$.
Applying the first Lemma, we choose $(i_0, j_0) \in \{i+1, \dots, m\} \times \{j+1, \dots, n\}$ such that $a_{i_0 j_0} = 1$, and

for any $(r, c) \in \{i, \dots, i_0 - 1\} \times \{j, \dots, j_0 - 1\}, \quad \sigma_{rc}(A) > \sigma_{rc}(C)$.

So

$$A[\{i, i_0\}; \{j, j_0\}] = \begin{bmatrix} 1 & * \\ * & 1 \end{bmatrix}.$$

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Then

$$A \prec_{\widehat{B}} C.$$

Case 2: $a_{i_0 j} = 1$.

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Applying Lemma 2, there is $(i_1, j_1) \in \{1, \dots, i_0 - 1\} \times \{1, \dots, j - 1\}$ such that $a_{i_1, j_1} = 1$ and

for any $(r, c) \in \{i_1, \dots, i_0 - 1\} \times \{j_1, \dots, j - 1\}$, $\quad \sigma_{r,c}(A) > \sigma_{r,c}(C)$.

We now consider three subcases.

Subcase 2.1: $i_1 = i$.

Then

$$A[\{i, i_0\}; \{j_1, j, j_0\}] = \begin{bmatrix} 1 & 1 & * \\ * & 1 & 1 \end{bmatrix}.$$

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Subcase 2.1: $i_1 = i$.

Then

$$A[\{i, i_0\}; \{j_1, j, j_0\}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus, $A[\{i, i_0\}; \{j_1, j_0\}] = I_2$ and

for any $(r, c) \in \{i, \dots, i_0 - 1\} \times \{j_1, \dots, j_0 - 1\}, \quad \sigma_{r,c}(A) > \sigma_{r,c}(C)$.

So

$$A \prec_{\widehat{B}} C.$$

Corollary

Let $R = (1, 1, \ldots, 1)$ and $S = (s_1, \ldots, s_n)$ be two nonincreasing positive integral vectors such that $\mathcal{A}(R, S) \neq \emptyset$. Then the Bruhat order and the Secondary Bruhat order coincide on $\mathcal{A}(R, S)$.

Proof:

Let $A, C \in \mathcal{A}(R, S)$ such that $A \preceq_B C$. Let D be the m-by-$n+1$ matrix such that the first column has all entries equal to one and removing the first column we have the matrix A. Similarly, let E be the m-by-$n+1$ matrix such that the first column has all entries equal to one and removing the first column we have the matrix C. Then D and E are matrices in $\mathcal{A}(U, V)$ with $U = (2, 2, \ldots, 2)$ and $V = (m, s_1, \ldots, s_n)$. Since $A \preceq_B C$ we get $D \preceq_B E$. By last theorem, $D \preceq_{\widehat{B}} E$. Thus, as the first columns of D and E have all entries equal to one, $A \preceq_{\widehat{B}} C$.

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