Shifted tableau crystals

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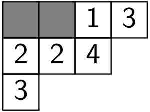
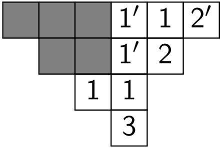
joint with Maria Gillespie (CSU),
Kevin Purbhoo (Waterloo)

"in Lisbon"
7 December 2020

Ordinary and shifted tableaux

- Semistandard (skew) tableaux  :
  - geometry of Grassmannians, rep theory of GL_n and S_n, combinatorics of symmetric functions, jeu de taquin.
- **Shifted** (skew) tableaux  :
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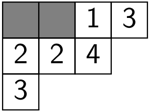
- Semistandard (skew) tableaux 

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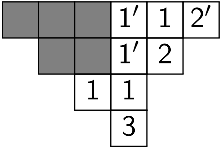
- **Shifted** (skew) tableaux 

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Ordinary and shifted tableaux

- Semistandard (skew) tableaux

		1	3
2	2	4	
3			

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- **Shifted** (skew) tableaux

			$1'$	1	$2'$
			$1'$	2	
		1	1		
			3		

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  - Several variants! Schur P-functions $\neq$ Schur Q-functions

    (For us: southwesternmost i/i' is always unprimed.)

From geometry to tableaux...

- Remarkable story connecting **geometry of curves**, **Schubert calculus**, **tableaux**

  [Shapiro–Shapiro, Eisenbud–Harris, Mukhin–Tarasov–Varchenko, Purbhoo, Speyer, Sottile, Halacheva–Kamnitzer–Rybnikov–Weekes, Osserman, White, Chan–López Martín–Pflueger–Teixidor i Bigas, Gillespie–L, **Rodrigues**...]

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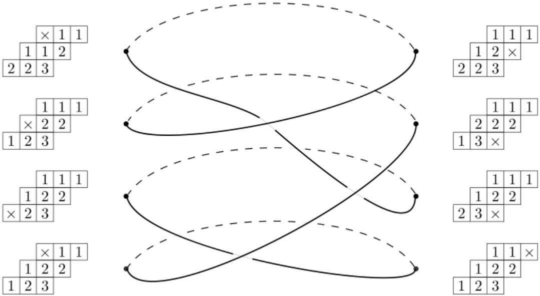
- Geometry story:
  - Define curves in $\mathrm{Gr}(k, n)$, $\mathrm{OG}(n, 2n + 1)$ by intersecting certain Schubert varieties



- Monodromy via (shifted) **tableaux** and **tableau algorithms**.

… to more tableaux

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Question

What are the natural coplactic operators on shifted tableaux?

Natural operations on tableaux – type A

Any coplactic operation is determined by its action on **rectified** tableaux.

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Tableaux of shape $\lambda = (5, 3)$, organized by weight:

1	1	1	1	1
2	2	2		

1	1	1	1	2
2	2	2		

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2	2	2		

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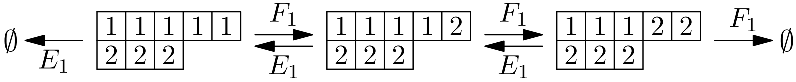

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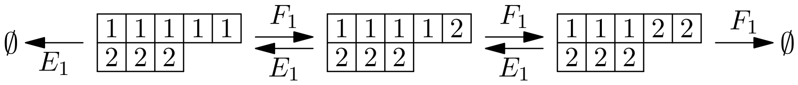
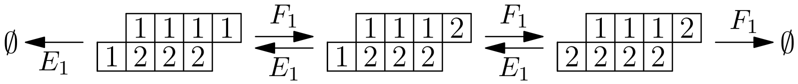


Action on general tableaux:



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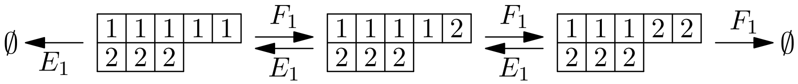

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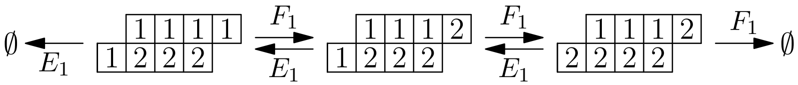

E_i, F_i: treat $i, i+1$ as $1, 2$.

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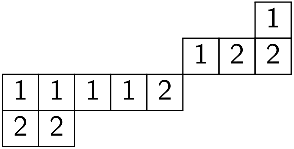

$$\mathrm{word}(T) = \begin{array}{ccccccccccc} 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ (& (&) &) &) &) & (&) & (& (&) \end{array}$$

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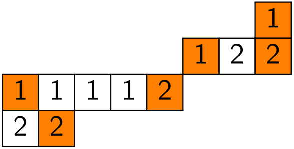

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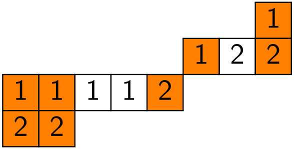

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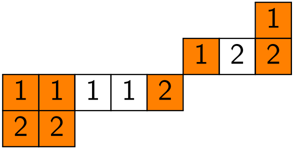

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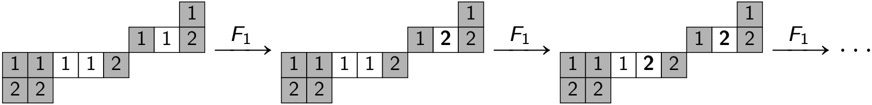
Result:



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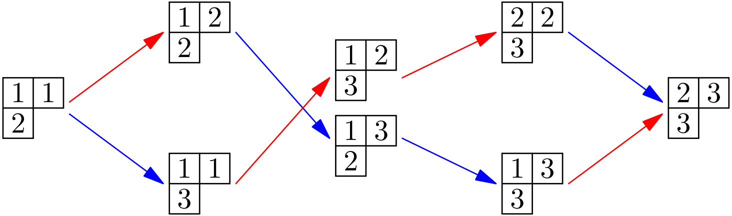

$F_1 = \textcolor{red}{\longrightarrow}$, $F_2 = \textcolor{blue}{\longrightarrow}$

(F_i : changes an $i \rightsquigarrow i{+}1$)

Uniquely determined JDT-invariant **graph structure** on tableaux.

Type B: *Two* operations on shifted Q-tableaux

Shifted tableaux of rectified shape $\sigma = (4, 1)$, by weight:

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1	1	1	1
	2		

1	1	1	2
	2		

1	1	1	$2'$
	2		

1	1	2	2
	2		

1	1	$2'$	2
	2		

1	$2'$	2	2
	2		

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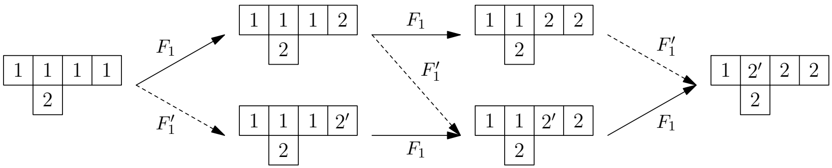

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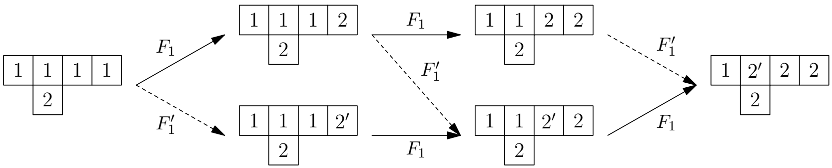

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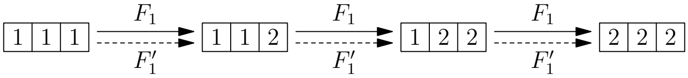

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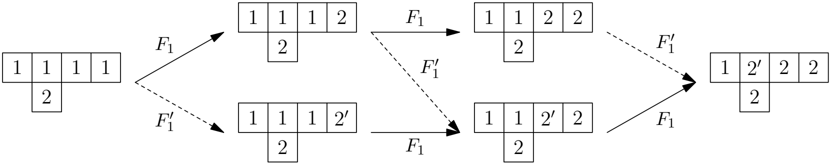

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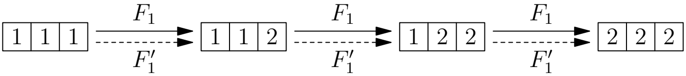

By coplacticity: unique operators $\xrightarrow{F_1}$, $\xrightarrow{F_1{}'}$ on all skew shifted tableaux.

Analog of a bracketing rule: F_i, F_i'

Theorem (Gillespie-L-Purbhoo '17)

There are direct definitions of F_i, F_i', depending only on $w = \mathrm{word}(T)$, via **first-quadrant lattice walks**.

- A 'doubled' bracketing rule!
- Allows monodromy computations in $\mathrm{OG}(n, 2n+1)$ [GLP19]

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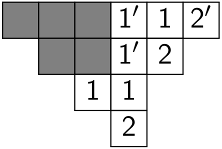
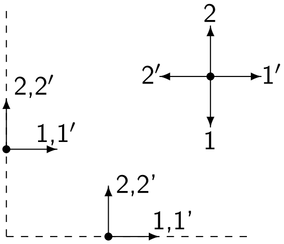
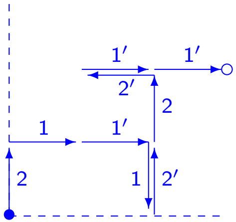


$: \mathrm{word}(T) = 2111'21'12'.$

First-quadrant lattice walks

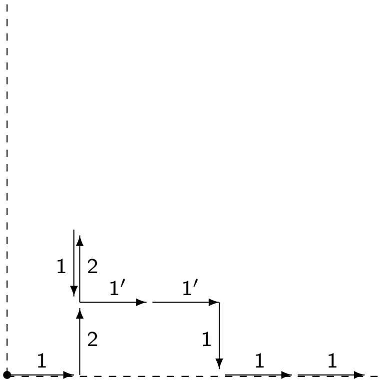




The **lattice walk** for $w = 211'12'22'1'1'$ ends at the point $(3, 2)$.

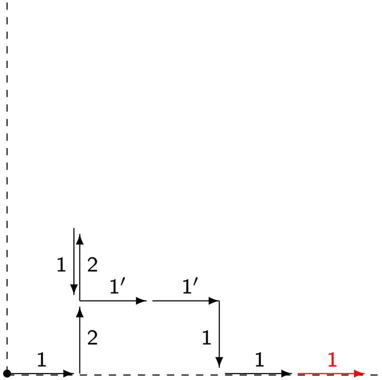
‘Cancellation’ away from the axes.

Example: repeated F_1, followed by one F_1':



Example: repeated F_1, followed by one F_1':

Action of F_1 on words: by transforming critical strings near axes.



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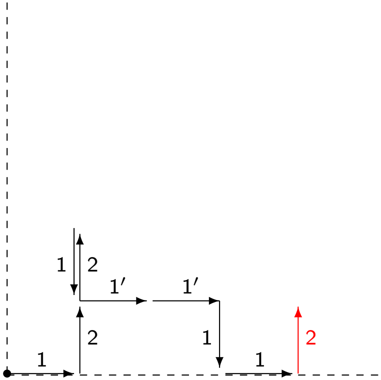

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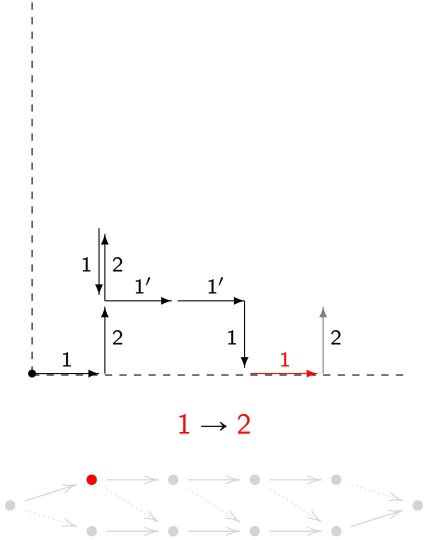

Example: repeated F_1, followed by one F'_1:

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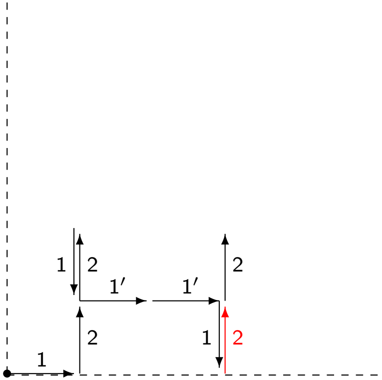

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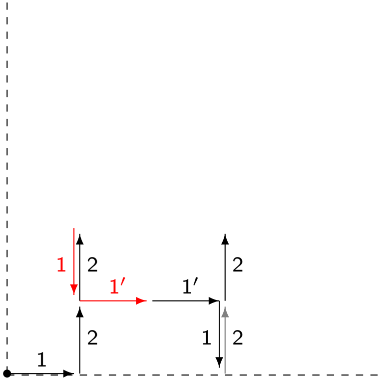

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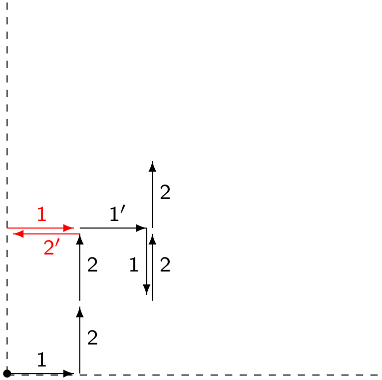

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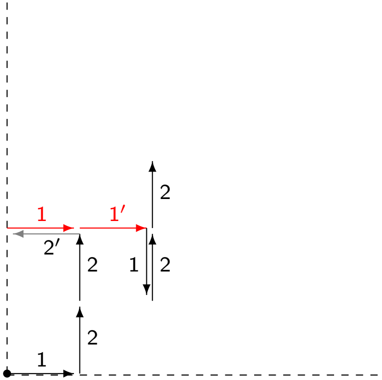

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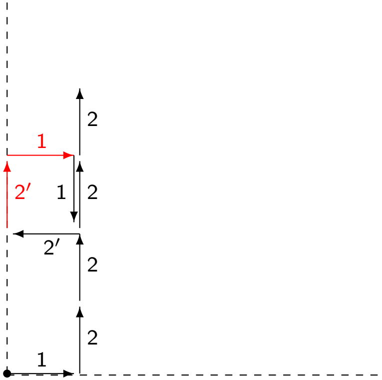

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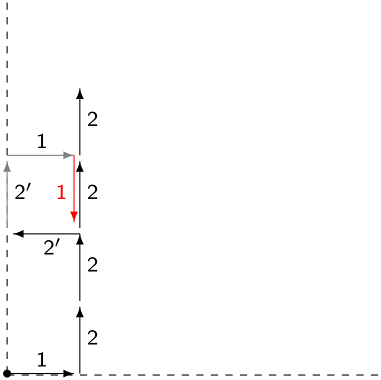

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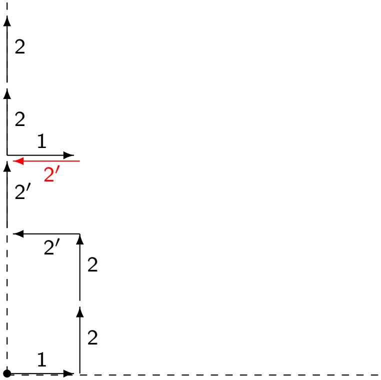

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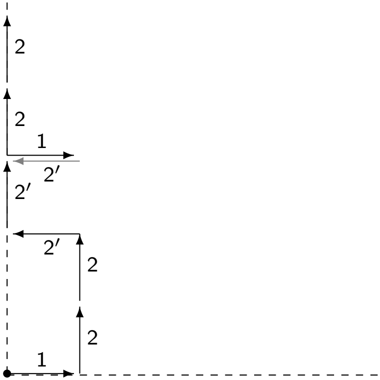
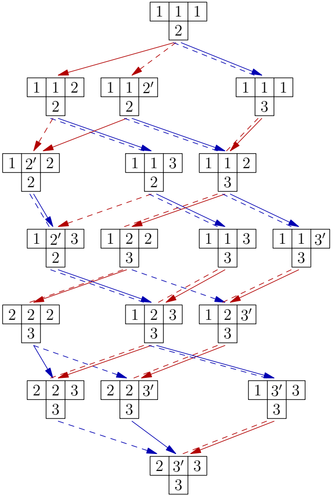
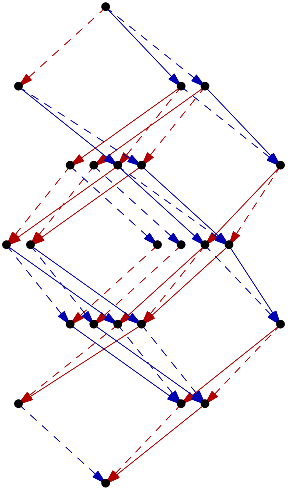


$$1 \to 2' \quad (F_1')$$

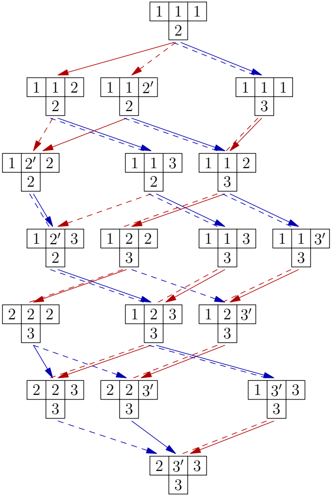
Example: the crystals $\mathcal{B}(\ydiagram{3,1+1}, 3)$ and $\mathcal{B}(\ydiagram{4,1+3,2+1}, 3)$





Legend: F_1 →, $F_1{}'$ ⇢, F_2 →, $F_2{}'$ ⇢

Features of shifted tableau crystals



Key features:

- Unique highest-weight element (type B LR tableau)
- Weighted characters are skew Schur Q-functions
- Connected components of $\mathrm{ShST}(\lambda/\mu, n)$ recover **skew LR rule** for Schur Q-functions,

$$\mathrm{ShST}(\lambda/\mu, n) \;\cong\; \bigsqcup_{\nu} \mathrm{ShST}(\nu, n)^{f^{\lambda}_{\nu,\mu}}$$

$$Q_{\lambda/\mu} \;=\; \sum_{\nu} f^{\lambda}_{\nu,\mu} Q_{\nu}.$$

- Cactus group action by *local reversals* (Rodrigues 2020)

Characterizing the graph structure

- F_i, F'_i: the **unique** lowering operators on shifted Q-tableaux, coplactic for shifted JDT.
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Theorem (Stembridge '03)

In type A, crystals are characterized by a short list of local graph-theoretic axioms (relating F_i, F_j).

Similar statement for shifted tableau crystals:

Theorem (Gillespie-L '18)

Shifted tableau crystals are characterized by a short list of local graph-theoretic axioms.

Four operators: $F_i, F_i', F_{i+1}, F_{i+1}' \rightsquigarrow$ 6 pairs.

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Theorem (Stembridge '03)

Local axioms in type A:

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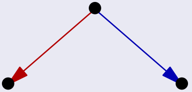

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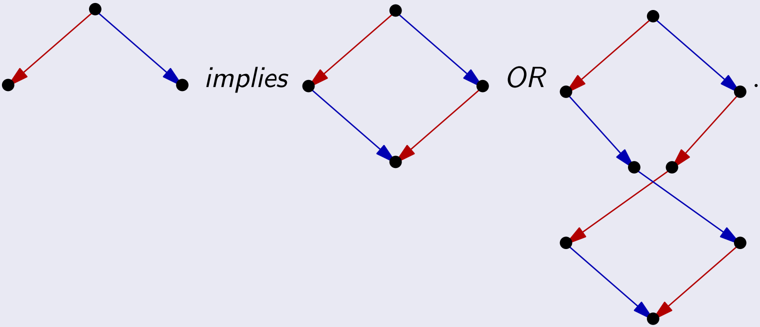

(depending on local data).

Axioms for shifted tableau crystals

Theorem (Gillespie-L '18)

Shifted Q-tableau crystals are characterized by a short list of local axioms (relating F_i, F_i', F_j, F_j').

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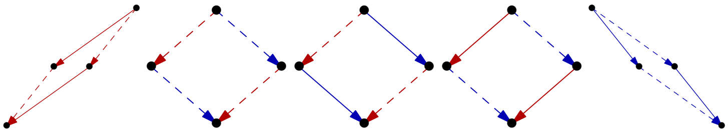
$$\{F_1', F_1\}, \{F_1', F_2'\}, \{F_1', F_2\}, \{F_1, F_2'\}, \{F_2', F_2\}$$



(Certain specific boundary-case exceptions.)

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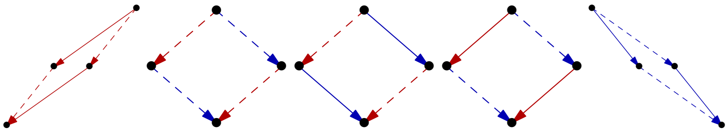
$$\{F'_1, F_1\}, \{F'_1, F'_2\}, \{F'_1, F_2\}, \{F_1, F'_2\}, \{F'_2, F_2\}$$



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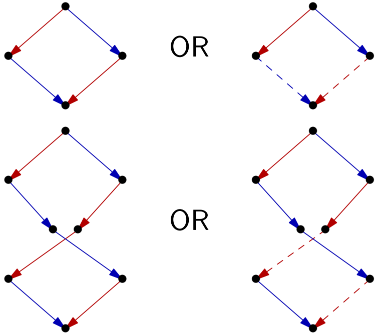
IDEA: Primed operator F'_i mostly "doubles" the crystal.

Axioms for shifted tableau crystals

The interesting pair: $\{F_1, F_2\}$. *Four* possibilities:



(depending on local data).

"Doubled" axioms from type A.

Using the axioms

Theorem (Gillespie-L '18)

Let G be a graph satisfying the local axioms for shifted tableau crystals. Then each connected component is $\cong \mathrm{ShST}(\sigma, n)$ for some σ.

Gives method to prove positivity of a generating function:

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Q: Can we use the axioms to compare:

- Shifted tableau crystals and $\mathfrak{q}(n)$ crystals? (Schur-P and Schur-Q duality?)
- Shifted tableau crystals and type A crystals by "flattening"?

Thank you!