Feasible regions meet pattern avoidance
The awaited 3rd part on feasible regions

Combinatorics Days

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Slides can be found in
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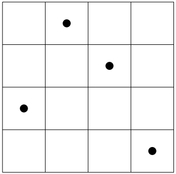
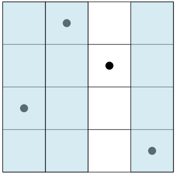
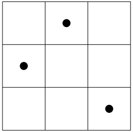
Patterns in permutations

A permutation π of size n is an arrangement on an $n \times n$ table:

$\pi =$  $= 2431$

The set of permutations of size n : $\mathcal{S}_n$
The set of all permutations : $\mathcal{S}$
Select a set I of columns of the square configuration of π and define the **restriction** $\pi|_I$. This is a permutation.

$\pi|_{\{1,2,4\}}$  $=$  $= 231$

Number of occurrences

We can count occurrences!
For permutations π, σ, we define the pattern number:

$$\mathrm{occ}(\pi, \sigma) = \#\{\text{occurrences of } \pi \text{ in } \sigma\}.$$

In this way we have

$$\mathrm{occ}(12, 4132) = 2,\ \mathrm{occ}(312, 4132) = 2,\ \mathrm{occ}(12, 12345) = 10$$

$$\text{and} \quad \mathrm{occ}(312, 3675421) = 0$$

For a fixed integer k, what are the possible values of $\left(\mathrm{occ}(\pi, \sigma)\, |\sigma|^{-|\pi|}\right)_{\pi \in \mathcal{S}_k}$ when $|\sigma|$ is big?

Plotting these relationships

$$\widetilde{\mathrm{occ}}(\pi,\tau)=\frac{\mathrm{occ}(\pi,\tau)}{\binom{|\tau|}{|\pi|}},\ \ \widetilde{\mathrm{occ}}_k(\tau)=(\widetilde{\mathrm{occ}}(\pi,\tau))_{\pi\in\mathcal{S}_k}\in\mathbb{R}^{\mathcal{S}_k}\,.$$

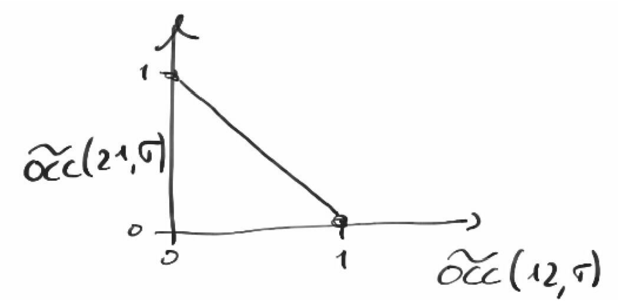


Figure: The interplay between proportion of occurrences of 12 and 21.

Feasible region - Classical patterns

For a fixed integer k, the corresponding feasible region (FReg) is defined as follows

$$F_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} \,|\, \exists\, \sigma^{(n)}, \widetilde{\mathrm{occ}}_k(\sigma^{(n)}) \to \vec{v}, |\sigma^{(n)}| \to \infty\}\,.$$

$F_{\leq k}$ - the FReg indexed by all permutations of size at most k
$F_{\mathcal{S}}$ - the FReg indexed by a set of permutations $\mathcal{S}$.

$F_{\{\pi\}}$ - an interval and is often studied in the context of *packing problems*.

Feasible region - Examples

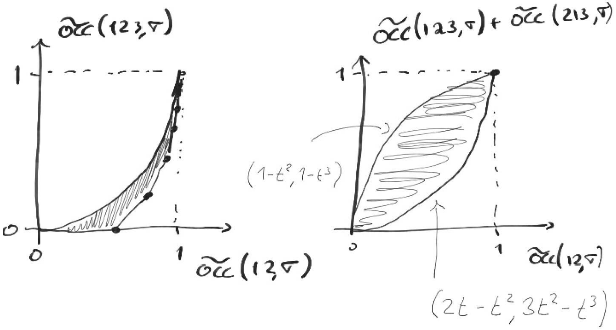


Figure: **Left:** The FReg comparing 12 and 123. **Right:** The FReg comparing patterns of 12 and patterns of 123 or 213 becomes a scalloped triangle. Feasible regions due to Kenyon, Kral, et.al. 2015.

Feasible region - The dimension problem

Theorem (Glebov, Hoppen, et.al. 2017)

The dimension of the feasible region $F_{\leq k}$ is at least the number of indecomposable permutations of size k.

Theorem (Vargas, 2014)

The feasible region $F_{\leq k}$ satisfies a set of algebraic equations indexed by the ***Lyndon permutations*** *of size up to k.*

Conjecture

The codimension of the feasible region $F_{\leq k}$ is precisely the number of ***Lyndon permutations*** *of size up to k.*

Consecutive occurrences

We now consider only occurrences that form **an interval.** For instance, taking $\sigma = 2413$, there are two distinct consecutive restrictions of σ of size three, namely 231 and 312.

$$\text{c-occ}(\pi, \tau) = \#\{I \text{ interval s.t. } \tau|_I = \pi\}\,.$$

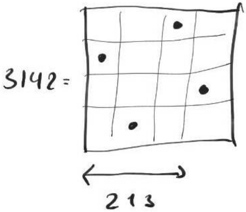


Figure: The permutation 3142, does not contain a consecutive occurrence of 231, but it does contain a consecutive occurrence of 213.

Consecutive occurrences

The number $\text{c-occ}(\pi, \tau)$ varies between 0 and $|\tau| - |\pi| + 1$. So we define

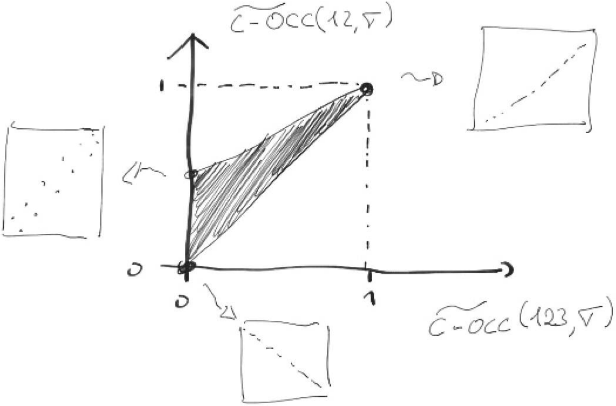
$$\widetilde{\text{c-occ}}(\pi, \tau) = \frac{\text{c-occ}(\pi, \tau)}{|\tau|}, \ \widetilde{\text{c-occ}}_k(\tau) = (\widetilde{\text{c-occ}}(\pi, \tau))_{\pi \in \mathcal{S}_k} \in \mathbb{R}^{\mathcal{S}_k} \ .$$

The “permuton” version for consecutive occurrences are called *shift-invariant random orders of* $\mathbb{Z}$, due to Borga(2018).

FRegIve

$$\mathcal{F}_k := \{\vec{v} \in \mathbb{R}^{\mathcal{S}_k} | \exists \sigma^{(n)}, \widetilde{\text{c-occ}}_k(\sigma^{(n)}) \to \vec{v}, |\sigma^n(n)| \to \infty\} \subseteq \mathbb{R}^{\mathcal{S}_k} .$$

This is a closed and convex region.



The overlap graph

Consider the case $k = 3$ and the permutation $\sigma = 2714365$.

$$2714365 \mapsto 231 - 312 - 132 - 213 - 132\,.$$

We can construct a graph from this:

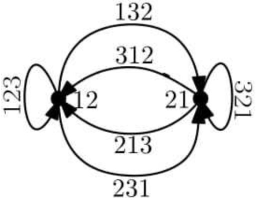


Figure: The overlap graph for $k = 3$

$\{$ permutations $\} \to \{$ paths in $\mathcal{O}v(k)\}$, is this map invertible?

The overlap graph - inverting a path

$$\omega = 2413 \to 4123 \to 1342 \to 2413\,.$$

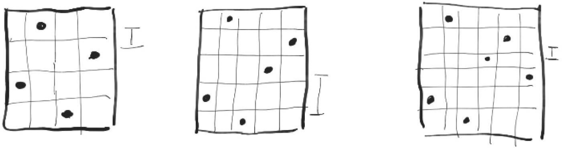


Figure: The construction of the path ω.

FRegIve is a cycle polytope

Theorem (Borga, P., 2019)

$$P(\mathcal{O}v(k)) = \mathcal{F}_k \, .$$

In particular, $\mathcal{F}$ *is a polytope with dimension* $k! - (k-1)!$.

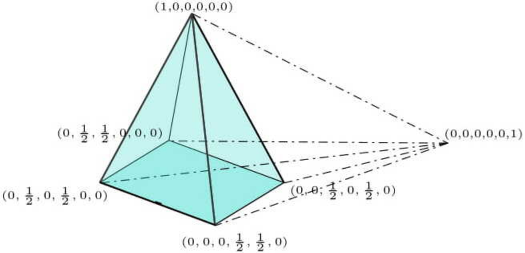


Figure: The feasible region of $k = 3$.

Avoiding set patterns - permutation classes

Let's introduce pattern avoidance in this problem!

$$\mathrm{Av}(\mathcal{P}) = \{\tau \in \mathcal{S} \,|\, \forall\, \pi \in \mathcal{P},\, \mathrm{occ}(\pi, \tau) = 0\}\,,$$

Let $\mathrm{Av}_k(\mathcal{P})$ be $\mathrm{Av}(\mathcal{P}) \cap \mathcal{S}_k$.

A set of the form $\mathrm{Av}(\mathcal{P}) \subseteq \mathcal{S}$ is called a **permutation class.**

Permutations classes are a world to be investigated!

Generating trees and ReFRegIve

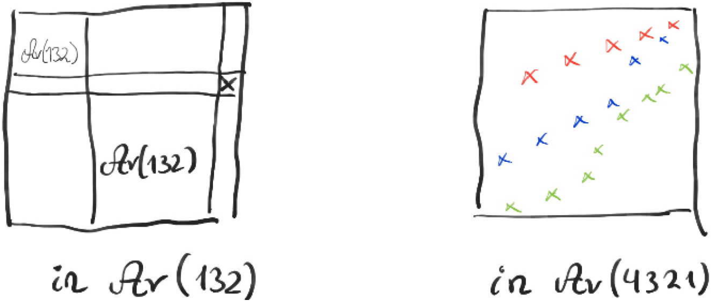


Figure: **Left:** the permutation class $\mathrm{Av}(132)$ is characterized by in inductive construction. **Right:** the permutation classes $\mathrm{Av}(n \cdots 1)$ are characterized by $n-1$ increasing monochromatic subsequences.

The feasible region (ReFRegIve) is:

$$\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})} := \{\vec{v} \in \mathbb{R}^{\mathrm{Av}_k(\mathcal{P})} \mid \exists\, \sigma^{(n)} \in \mathrm{Av}(\mathcal{P}) \text{ with } \widetilde{\text{c-occ}}(\sigma^{(n)}) \to \vec{v}\}\,.$$

Does anyone read these titles?

ReFRegIve is still a closed set. **Convexity on ReFRegIve**?
Example: if $\mathcal{P} = \{132, 312, 231, 213\}$, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is a set with only two points.

Proposition

If $\mathcal{P}$ is a singleton, then $\mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$ is convex.

{ permutations in $\mathrm{Av}(\mathcal{P})\} \to \{$ paths in $\mathcal{O}v(k)$ avoiding $\mathcal{P}$ }

Thus, $P(\mathcal{O}v(k)|_{\mathrm{Av}(\mathcal{P})}) \subseteq \mathcal{F}_k^{\mathrm{Av}(\mathcal{P})}$.

Example of path inversion - 132

On the case 132, can we always invert such paths? Example:

$$\omega = 123 \to 231 \to 321 \to 213\,.$$

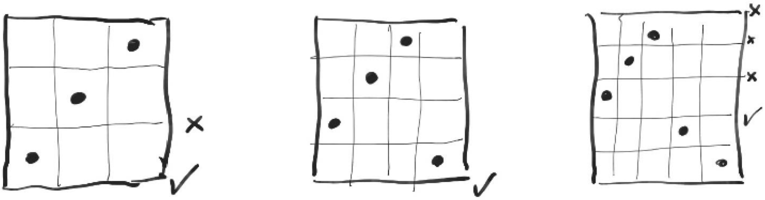


Figure: The construction of a permutation corresponding to the path ω.

The upshot - 132

$$\mathcal{F}_k^{\mathrm{Av}(132)} = P(\mathcal{O}v(k)|_{\mathrm{Av}_k(132)}) \text{ and } \dim \mathcal{F}_k^{\mathrm{Av}(132)} = C_k - C_{k-1}\,.$$

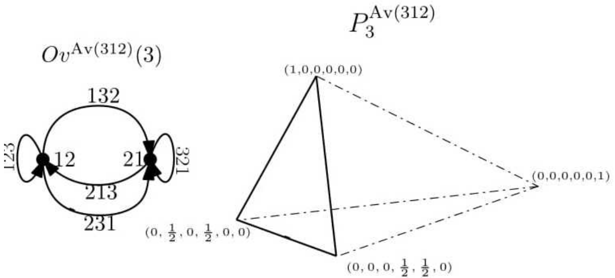


Figure: **Left:** The restricted overlap graph for $\mathcal{P} = \{312\}$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{312\}$.

The overlap graph - 321

On the case 321, can we always invert such paths? Example:

$$\omega = 312 \to 123 \to 231\,.$$

Let's add colours to the path, in such a way that each color is a monotone sequence:

$$\omega = {\color{red}3}{\color{blue}12} \to {\color{blue}123} \to {\color{blue}23}1\,.$$

The coloured overlap graph - 321

On the other hand, a valid sequence would be, for instance

$$\omega = 312 \to 123 \to 123 \to 132\,.$$

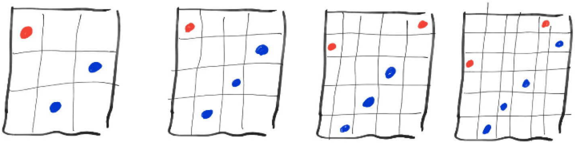


Figure: The construction of a permutation corresponding to the corrected path ω.

The coloured overlap graph - 321

Let's add colours to the overlap graph itself and call it $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(k)$

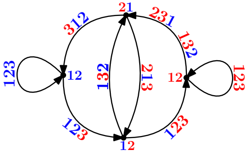


Figure: The overlap graph for $k = 3$ adapted to $\mathcal{P} = \{321\}$, where now we include all possible colouring of each edge.

Theorem (Borga, P. 2020)

$$\mathcal{F}_k^{\mathrm{Av}(n\cdots 1)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(n\cdots 1)}(k)))\,,$$

$$\dim \mathcal{F}_k^{\mathrm{Av}(n\cdots 1)} = |\,\mathrm{Av}_k(n\cdots 1)| - |\,\mathrm{Av}_{k-1}(n\cdots 1)|\,.$$

The restricted feasible region - 321

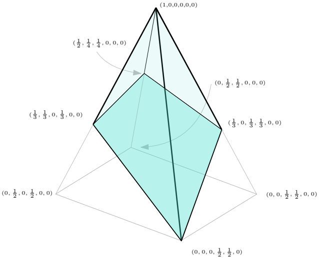


Figure: **Left:** $P(\mathcal{O}v(3))$. **Right:** The restricted feasible region for $k = 3$ and $\mathcal{P} = \{321\}$, overlaid with $P(\mathcal{O}v(3)|_{\mathrm{Av}_3(321)})$.

Future work

- Other permutation classes are also given by generating trees. We believe that any such permutation class will have a direct description of the feasible region, and that we can totally describe all the extremal points.
- The dimension conjecture: if $\mathcal{P}$ has only one pattern, then

$$\dim \mathcal{F}_k^{\mathrm{Av}_k(\mathcal{P})} = |\,\mathrm{Av}_k(\mathcal{P})| - |\,\mathrm{Av}_{k-1}(\mathcal{P})|\,.$$

- The other dimension conjecture on the classical FReg.
- What is the volume of all these regions? Related with triangulations.
- Combinatorial structure of cycle polytopes.

The end