Diagrammatics of generalized majorization

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Classical majorization of partitions

Comparison of partitions of integers (= finite non-increasing sequence of integers.)

Hardy-Littlewood-Pólya majorization for two partitions:

$$\mathbf{g} \prec \mathbf{b}.$$

For $\mathbf{g} = (g_1, \ldots, g_s)$, and $\mathbf{b} = (b_1, \ldots, b_s)$:

$$\begin{aligned} \sum_{i=1}^{j} g_i &\leq \sum_{i=1}^{j} b_i, \qquad j = 1, \ldots, s-1, \\ \sum_{i=1}^{s} g_i &= \sum_{i=1}^{s} b_i. \end{aligned}$$

Generalized majorization

Extension to three partitions of integers:

$$\begin{aligned}\mathbf{g} &= (g_1, \ldots, g_{m+s}),\\ \mathbf{c} &= (c_1, \ldots, c_m),\\ \mathbf{b} &= (b_1, \ldots, b_s)\end{aligned}$$

with $|g| = |c| + |b|$.

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Definition

$$\begin{aligned} & c_i \geq g_{i+s}, && i = 1, \ldots, m, \\ & \sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} c_i \leq \sum_{i=1}^{j} b_i, && j = 1, \ldots, s-1, \end{aligned}$$

where $h_j := \min\{i | c_{i-j+1} < g_i\}, \quad j = 1, \ldots, s$, then we say that

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

Generalized majorization – motivation

- For $\mathbf{c} = \emptyset$, reduces to classical majorization.

- Completion of matrices and matrix pencils.

Two of three partitions are column (or row) minimal indices of the two pencils, while the third one depends on invariants factors and polynomials paths.

- Also, appears in Representation theory of Quivers.

Diagrammatics of generalized majorization

Diagrammatic notation: We denote

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b})$$

by

$$\begin{tikzcd} \mathbf{g} \\ \mathbf{c} \arrow[u, "\mathbf{b}"'] \end{tikzcd}$$

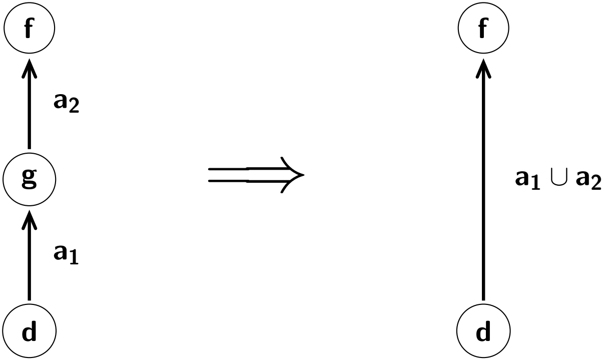
Diagrammatics of generalized majorization

Transitivity-like property [M.Dodig, M. Stosic, El. J. Comb., 2010]

$$\begin{tikzcd} \mathbf{f} \\ \mathbf{g} \arrow[u, "\mathbf{a_2}"'] \\ \mathbf{d} \arrow[u, "\mathbf{a_1}"'] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathbf{f} \\ \\ \mathbf{d} \arrow[uu, "\mathbf{a_1} \cup \mathbf{a_2}"'] \end{tikzcd}$$

Diagrammatics of generalized majorization

Transitivity-like property [M.Dodig, M. Stosic, El. J. Comb., 2010]



This type of properties have "linear" diagrammatical description

Double generalized majorization

Problem

For given partitions $\mathbf{d}, \mathbf{g}, \mathbf{a}, \mathbf{b}$*, find necessary and sufficient conditions for the existence of a partition* $\mathbf{f}$*, such that*

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \textit{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}).$$

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Very important in applications in matrix pencil completion and representation theory of Kronecker quiver.

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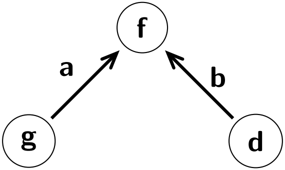
Very important in applications in matrix pencil completion and representation theory of Kronecker quiver.

Very complicated. Explicit necessary and sufficient conditions have been obtained...but are quite involved.

In pictures, the question is when there exists **f** such that:



Dual double generalized majorization

Problem

Find necessary and sufficient conditions for the existence of a partition $\mathbf{c}$*, such that*

$$\mathbf{g} \prec' (\mathbf{c}, \mathbf{b}) \quad and \quad \mathbf{d} \prec' (\mathbf{c}, \mathbf{a}).$$

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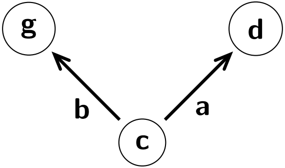

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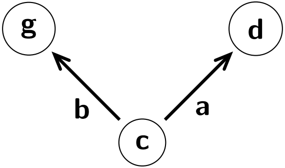

Equally important for application.
Very similar to double generalized majorization, but is different from double generalized majorization and still NO complete explicit solution.

Main result: completion of square

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Theorem (M. Dodig, M. Stosic, 2021)

A solution of dual double generalized majorization problem gives a solution of double generalized majorization problem.

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A solution of dual double generalized majorization problem gives a solution of double generalized majorization problem.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, $\mathbf{g}$, be given partitions.
If there exists a partition $\mathbf{c}$ such that

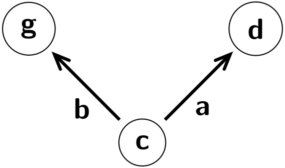
$$\mathbf{d} \prec' (\mathbf{c}, \mathbf{a}) \quad \text{and} \quad \mathbf{g} \prec' (\mathbf{c}, \mathbf{b}).$$

then there exists a partition $\mathbf{f}$ such that

$$\mathbf{f} \prec' (\mathbf{d}, \mathbf{b}) \quad \text{and} \quad \mathbf{f} \prec' (\mathbf{g}, \mathbf{a}).$$

Main result: completion of square

Diagrammatically: completion of a square



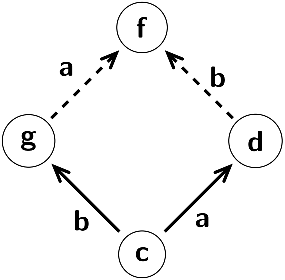
Main result: completion of square

Diagrammatically: completion of a square

$$\begin{tikzcd}
 & \mathbf{f} & \\
\mathbf{g} \arrow[ur, dashed, "\mathbf{a}"] & & \mathbf{d} \arrow[ul, dashed, "\mathbf{b}"'] \\
 & \mathbf{c} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{a}"'] &
\end{tikzcd}$$

Main result: completion of square

Diagrammatically: completion of a square



NOTE: Converse does not hold.

Playing in the plane

Some corollaries of combining the results:
Let $\mathbf{c}$, $\mathbf{u}$, $\mathbf{w}$, $\mathbf{g}$, $\mathbf{d}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{b_1}$ and $\mathbf{b_2}$ be partitions such that

$$\mathbf{u} \prec' (\mathbf{g}, \mathbf{a_1}), \quad \mathbf{u} \prec' (\mathbf{c}, \mathbf{b_1}), \quad \mathbf{w} \prec' (\mathbf{c}, \mathbf{a_2}), \quad \mathbf{w} \prec' (\mathbf{d}, \mathbf{b_2}),$$

i.e.

$$\begin{tikzcd}
 & \mathbf{u} & & \mathbf{w} & \\
\mathbf{g} \arrow[ur, "\mathbf{a_1}"] & & \mathbf{c} \arrow[ul, "\mathbf{b_1}"'] \arrow[ur, "\mathbf{a_2}"'] & & \mathbf{d} \arrow[ul, "\mathbf{b_2}"']
\end{tikzcd}$$

Playing in plane

Completion of (middle) square:

$$\begin{tikzcd}
 & \mathbf{u} & & \mathbf{w} & \\
\mathbf{g} \arrow[ur, "\mathbf{a_1}"] & & \mathbf{c} \arrow[ul, "\mathbf{b_1}"'] \arrow[ur, "\mathbf{a_2}"'] & & \mathbf{d} \arrow[ul, "\mathbf{b_2}"']
\end{tikzcd}$$

Playing in plane

Completion of (middle) square:

$$
\begin{tikzcd}
 & & \mathbf{f} & & \\
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\end{tikzcd}
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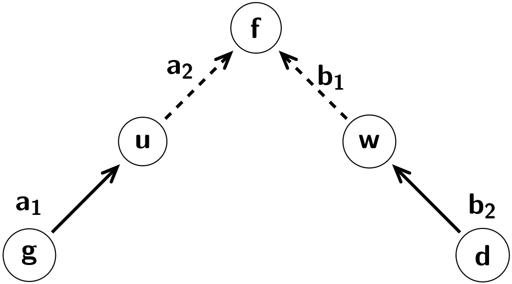

Playing in plane

Completion of (middle) square:

$$\begin{tikzcd}
 & & \mathbf{f} & & \\
 & \mathbf{u} \arrow[ur, dashed, "\mathbf{a_2}"] & & \mathbf{w} \arrow[ul, dashed, "\mathbf{b_1}"'] & \\
\mathbf{g} \arrow[ur, "\mathbf{a_1}"] & & & & \mathbf{d} \arrow[ul, "\mathbf{b_2}"']
\end{tikzcd}$$

Finally, by transitivity, such **f** satisfies:

$$\begin{tikzcd}
 & \mathbf{f} & \\
\mathbf{g} \arrow[ur, "\mathbf{a} = \mathbf{a_1} \cup \mathbf{a_2}"] & & \mathbf{d} \arrow[ul, "\mathbf{b} = \mathbf{b_1} \cup \mathbf{b_2}"']
\end{tikzcd}$$

Thank you for your attention !