Pak-Stanley labeling of the Ish arrangement

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Outline

Hyperplane arrangements

Parking functions

Pak-Stanley labeling

The Ish arrangement

Hyperplane arrangements

n-dimensional Coxeter arrangement or braid arrangement:

$$\mathrm{Cox}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0\}$$

n-dimensional Shi arrangement (Shi, 1986):

$$\mathrm{Shi}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0, x_i - x_j = 1\}$$

n-dimensional Ish arrangement (Armstrong, 2012):

$$\mathrm{Ish}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0, x_1 - x_j = i\}$$

Shi_3 and Ish_3

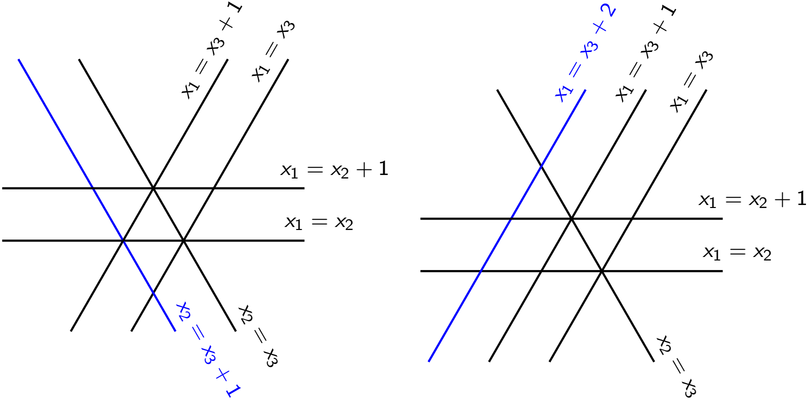

Figure: Shi_3 (left) and Ish_3 (right)

Theorem (Athanasiadis, 1996)

The characteristic polynomial of Cox_n *is* $\chi(\mathrm{Cox}_n, q) = q(q-1)\cdots(q-n+1)$.

Theorem (Athanasiadis, 1996)

The characteristic polynomial of Shi_n *is* $\chi(\mathrm{Shi}_n, q) = q(q-n)^{n-1}$.

Theorem (Armstrong & Rhoades, 2012)

The characteristic polynomial of Ish_n *is* $\chi(\mathrm{Ish}_n, q) = q(q-n)^{n-1}$.

hyperplane arrangement	# regions	# rel. bounded regions
Cox_n	$n!$	0
Shi_n	$(n+1)^{n-1}$	$(n-1)^{n-1}$
Ish_n	$(n+1)^{n-1}$	$(n-1)^{n-1}$

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Parking functions

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Assume n drivers want to park on a one-way street with n parking spaces.
ith driver prefers space a_i. If a_i is occupied, i takes the next available space.
$\mathbf{a} = (a_1, \ldots, a_n)$ is a parking function of length n if all cars can park.

Example

(3,1,1) is a parking function of length 3
_ _ _ $\rightarrow$ _ _ 1 $\rightarrow$ 2 _ 1 $\rightarrow$ 2 3 1
(1,3,3) is not a parking function of length 3
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$$\begin{aligned}\mathrm{PF}_n =& \{\text{parking funtions of length } n\}\\ \mathrm{PF}_2 =& \{11, 12, 21\}\\ \mathrm{PF}_3 =& \{111, 112, 113, 121, 122, 123, 131, 132,\\ & 211, 212, 213, 221, 231,\\ & 311, 312, 321\}\end{aligned}$$

$\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is a parking function of length n iff

$$|\mathbf{a}^{-1}([i])| = |\{j \in [n] \mid a_j \leq i\}| \geq i, \text{ for every } i \in [n].$$

Note that

$$\mathbf{a}^{-1}([i]) = \{\text{drivers who want to park in the first } i \text{ places}\}.$$

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A parking function $\mathbf{a} \in \mathbb{N}^n$ of length n is prime iff it remains a parking function (of length $n-1$) when we remove a 1.

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$$|\,\mathrm{PF}_n\,| = (n+1)^{n-1} \qquad \text{and} \qquad |\,\mathrm{PF}'_n\,| = (n-1)^{n-1}$$

Pak-Stanley labeling

In the 90's Pak and Stanley introduced a bijection between the regions of the Shi arrangement and parking functions which triggered a variety of research projects in several directions.

Among these projects is the determination of the sets of labels of the regions of other arrangements labeled using the same rules and an easy way to return a region from its label.

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Pak-Stanley labeling (adapted)

Let $R_0 = \{\mathbf{x} \in \mathbb{R}^n \mid x_n + 1 > x_1 > \cdots > x_n\}$ be the region bounded by the hyperplanes of equation $x_i - x_{i+1} = 0$, for $i \in [n-1]$, and $x_1 - x_n = 1$.

$\ell(R_0) := \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

Let R_1 and R_2 be two regions separated by a unique hyperplane H, of equation $x_i - x_j = k$, such that R_0 and R_1 are on the same side of H. Then

$$\ell(R_2) = \begin{cases} \ell(R_1) + e_i & \text{if } k \leq 0, \\ \ell(R_1) + e_j & \text{if } k > 0, \end{cases}$$

where $e_i = (0, \ldots, 0, \underbrace{1}_{i\text{th pos.}}, 0, \ldots, 0)$

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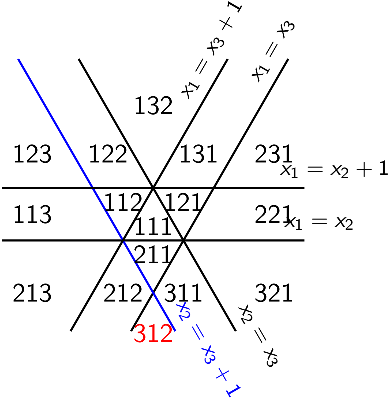

Figure: Pak-Stanley labelings of Shi_3 (left) and Ish_3 (right)

The Pak-Stanley labels of the regions of the Shi arrangement are the parking functions and the labels of the relatively bounded regions of the Shi arrangement are the prime parking functions.

Definition (D. and Guedes de Oliveira, 2021)

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The *0-center* of $\mathbf{a}$, $Z_0(\mathbf{a})$, is the largest subset $X = \{x_1, \ldots, x_q\}$ of $[n]$ such that if $1 \leq x_q < x_{q-1} < \cdots < x_1 \leq n$ then

$$(a_{x_q}, a_{x_{q-1}}, \ldots, a_{x_2}, a_{x_1}) \preceq (q, q-1, \ldots, 2, 1)$$

where $\preceq$ denotes the product order or componentwise order. More generally, if $p \in \mathbb{Z}$, the *p-center* of $\mathbf{a}$, $Z_p(\mathbf{a})$, is the largest subset $X = \{x_1, \ldots, x_q\}$ of $[n]$ such that if $1 \leq x_q < x_{q-1} < \cdots < x_1 \leq n$ then

$$(a_{x_q}, a_{x_{q-1}}, \ldots, a_{x_2}, a_{x_1}) \preceq (p+q, p+q-1, \ldots, p+2, p+1).$$

Example

Let $\mathbf{a} = 1352$.

$\mathbf{a} = 1352$ ($a_1 = 1 \preceq 1$ and there is no $a_{i_1} a_{i_2} \preceq 21$)

$Z_0(\mathbf{a}) = \{1\}$

$\mathbf{a} = 1352$ ($a_1 a_2 a_4 = 132 \preceq 432$ and $\mathbf{a} \not\preceq 5432$)

$Z_1(\mathbf{a}) = \{1, 2, 4\}$

$\mathbf{a} = 1352$ ($a_1 a_2 a_4 = 132 \preceq 543$ and $\mathbf{a} \not\preceq 6543$)

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$\mathbf{a} = 1352$ ($\mathbf{a} \preceq 7654$)

$Z_3(\mathbf{a}) = \{1, 2, 3, 4\}$

($Z_p(\mathbf{a}) = \varnothing$, if $p < 0$, and $Z_p(\mathbf{a}) = [4]$, if $p > 3$.)

Let $\pi \in \mathcal{S}_n$. The *inversion table* of π, $I(\pi)$, is defined by

$$I(\pi)_i = |\{j > i \mid \pi^{-1}(j) < \pi^{-1}(i)\}|, \quad \text{for every } i \in [n].$$

For every $\mathbf{a} \in \mathbb{N}^n$, let $\mathbf{p}(\mathbf{a}), \mathbf{q}(\mathbf{a}) \in (\mathbb{N} \cup \{0\})^n$ be defined by

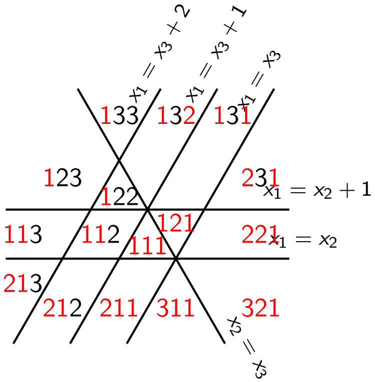
$$\mathbf{p}(\mathbf{a})_i := \min\{j \in \mathbb{N} \cup \{0\} \mid i \in Z_j(\mathbf{a})\}$$

and

$$\mathbf{q}(\mathbf{a})_i := \min\{\mathbf{p}(\mathbf{a})_i, i - 1\},$$

for every $i \in [n]$.

The Ish arrangement



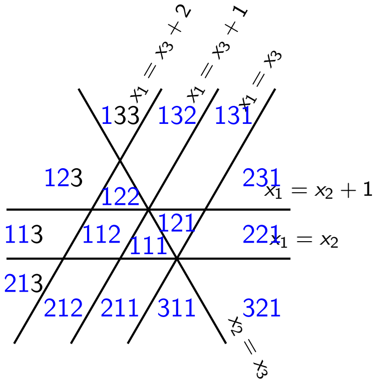


Figure: The "0-centers" (left) and the "1-centers" (right)

Theorem (D. and Guedes de Oliveira, 2018)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ *is the Pak-Stanley label of a region of* Ish_n *if and only if* $\mathbf{a} \in [n]^n$ *and* $1 \in Z_0(\mathbf{a})$. *This labeling is bijective.*

Example

$$\begin{aligned}\mathrm{IPF}_n =& \{\text{Pak-Stanley labels of the regions of } \mathrm{Ish}_n\} \\ \mathrm{IPF}_3 =& \{111, 112, 113, 121, 122, 123, 131, 132, 133, \\ & 211, 212, 213, 221, 231, \\ & 311, 321\}\end{aligned}$$

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Theorem (D.)

Given $\mathbf{a} \in \mathbb{N}^n$*,* $\mathbf{a}$ *is the Pak-Stanley label of a relatively bounded region of* Ish_n *if and only if* $\mathbf{a} \in [1] \times [n-1]^{n-1}$*.*

Example

$$\mathrm{IPF}'_n = \{\text{Pak-Stanley labels of the rel. bounded regions of } \mathrm{Ish}_n\}$$
$$\mathrm{IPF}'_3 = \{111, 112, 121, 122\}$$

Theorem (D.)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ *is the Pak-Stanley label of a relatively bounded region of* Ish_n *if and only if* $\mathbf{a} \in [1] \times [n-1]^{n-1}$.

Example

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Theorem (D.)

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Example

$$\begin{aligned}\mathrm{IPF}'_n =& \{\text{Pak-Stanley labels of the rel. bounded regions of } \mathrm{Ish}_n\}\\ \mathrm{IPF}'_3 =& \{111, 112, 121, 122\}\end{aligned}$$

Theorem (D.)

Let R be a region of Ish_n *and* $1 \leq i < j \leq n$. *Then* R_0 *and* R *are separated by the hyperplane of equation* $x_1 = x_j + i$ *if and only if* $j \notin Z_{i-1}(\ell(R))$.

Theorem (D.)

Let R be a region of Ish_n, $\mathbf{a} := \ell(R)$ *and* $\pi \in \mathcal{S}_n$. *Then*

$$R \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$$

if and only if

$$\mathbf{a} - \mathbf{q}(\mathbf{a}) = \mathbf{1} + I(\pi).$$

$\mathbf{1} + I(\pi)$ is the minimum label among the labels of the regions of Ish_n contained in $\{\mathbf{x} \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$.

Example

Let R be the region of Ish_n such that $\mathbf{a} = \ell(R) = 1424$. Then $Z_0 = \{1\}$, $Z_1 = \{1,3\}$, $Z_2 = \{1,2,3\}$, $Z_3 = \{1,2,3,4\}$. It follows that $\mathbf{p}(\mathbf{a}) = 0213$ and $\mathbf{q}(\mathbf{a}) = 0213 \wedge 0123 = 0113$. Hence $1 + I(\pi) = \mathbf{a} - \mathbf{q}(\mathbf{a}) = 1311$ and $\pi = 1342$.

$\_\ \_\ \_\ \_ \to 1\ \_\ \_\ \_ \to 1\ \_\ \_\ 2 \to 1\ 3\ \_\ 2 \to 1\ 3\ 4\ 2$

Finally,

$$R = \{\mathbf{x} \in \mathbb{R}^4 \mid \overbrace{x_1 > x_3 > x_4 > x_2}^{\pi=1342}, \underbrace{x_1 > x_2 + 1}_{q_2=1}, \underbrace{x_3 + 1 < x_1 < x_3 + 2}_{q_3=1}, \underbrace{x_1 > x_4 + 3}_{q_4=3}\}.$$

Open problem: The Pak-Stanley labeling of both the m-Shi arrangements and m-Catalan arrangements is bijective. Is there a family of "m-Ish arrangements" for which the Pak-Stanley is bijective? What are the Pak-Stanley labels of the regions of the "m-Ish arrangements"?

Thank you for your attention!

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