The hidden forest in powers of differential operators

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11th Combinatorics Days

Precedents

E. Briand, S. Lopes, M. Rosas

- Let $h(x)$ be a smooth function and let $\partial = \frac{d}{dx}$.
- What is the normally ordered form of the operator $(h\partial)^n$, $n \in \mathbb{N}_0$?

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$$(h\partial)^2 = (h\partial)(h\partial) = 1h\partial(h)\partial + 1h^2\partial^2$$

$$(h\partial)^3 = (h\partial)(h\partial(h)\partial + h^2\partial^2) =$$
$$1h(\partial(h))^2\partial + 1h^2\partial^2(h)\partial + 3h^2\partial(h)\partial^2 + 1h^3\partial^3$$

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$$\partial_q(ab) = \partial_q(a)b + \sigma_q(a)\partial_q(b)$$

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Gaussian Coefficients

Definition

Binomial Coefficient

$\binom{m}{n} \in \mathbb{N}_0$ represents the number of binary sequences of length m where n digits are equal to 1.

Gaussian Coefficient

$\binom{m}{n}_q \in \mathbb{Z}[q]$ is such that the coefficient of q^k represents the number of binary sequences of length m where n digits are equal to 1 with k **inversions**, where an inversion is a pair of digits in the sequence such that the leftmost one is 1 and the other is 0.

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Essence: The gaussian coefficients are a **detailed version** of the binomial coefficients, taking in account the notion of **inversions**.

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- If $m = 3$, $n = 2$, there are 3 sequences:

$$011,\ 101,\ 110$$

- Thus, $\binom{3}{2} = 3$.

- The first one has 0 inversions, the second one has 1 inversion, and the third one has 2 inversions:

$$\binom{3}{2}_q = 1 \cdot q^0 + 1 \cdot q^1 + 1 \cdot q^2 = 1 + q + q^2.$$

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Definitions

Given $n \in \mathbb{N}_0$ e $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{N}_0^n$, we have the following definitions:

- $|\lambda| := n$ is the **length** of λ;
- $s_k^\lambda := \sum_{i=0}^{k} \lambda_i$, $\forall k \geq 0$; also, we define $s_{-1}^\lambda := 0$;
- We write
$$\lambda \vDash S$$
when S is the sum of the entries of λ, and we say that λ is a **pseudo-composition** of S; also, we define $S(\lambda) := S$;
- If $s_k^\lambda \leq k$, $\forall k \geq 0$, we say that λ is **normal**.

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Normal pseudo-compositions

Example

Let $n = 5$ and let $\lambda = (0, 0, 2, 0, 2)$. Then:

- λ has length 5: $|\lambda| = 5$;
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    - $s^{\lambda}_{-1} = 0$;
    - $s^{\lambda}_{0} = 0$;
    - $s^{\lambda}_{1} = 0 + 0 = 0$;
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Γ sets

Definition

We define Γ_n as the set of all normal pseudo-compositions of length n.

Normally ordered form of $(h\partial_q)^n$

Theorem

Let n be a non-negative integer. Then,

$$(h\partial_q)^n = \sum_{\lambda \in \Gamma_n} c_\lambda(q) \cdot h^{[\lambda]},$$

where

$$h^{[\lambda]} = \left(\prod_{k=0}^{|\lambda|-1} \sigma^{k-s_k^\lambda}(\partial_q^{\lambda_k}(h)) \right) \partial_q^{n-S(\lambda)}$$

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Definition

Given $n \in \mathbb{N}_0$ and $\lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{N}_0^n$, we say that λ is *d*-**normal** if $s_k^\lambda \leq dk$, $\forall k \geq 0$.

Definition

We define Γ_n^d as the set of all d-normal pseudo-compositions of length n.

Theorem

Let n be a non-negative integer and let d be a positive integer.

Then,

$$(h\partial_q^d)^n = \sum_{\lambda \in \Gamma_n^d} c_\lambda^d(q) \cdot h^{[\lambda]},$$

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$$h^{[\lambda]} = \left(\prod_{k=0}^{|\lambda|-1} \sigma^{kd - s_k^\lambda} (\partial_q^{\lambda_k}(h)) \right) \partial_q^{n-S(\lambda)}$$

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Combinatorial study of the coefficients $c_\lambda(q)$

How many normal pseudo-compositions are there?

Proposition

Let n be a non-negative integer. Then,

$$|\Gamma_n| = C_n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.

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Catalan numbers

- Closed Formula:

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- Combinatorial Interpretation: C_n = number of **Dyck Paths** on an $n \times n$ board.

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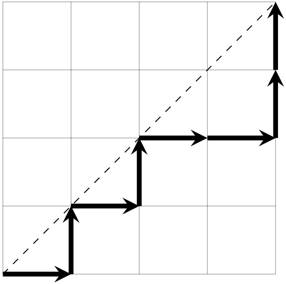

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An **increasing tree** with $n+1$ vertices is a rooted tree with vertices $0, \ldots, n$ such that $\text{father}(i) < i$, $\forall i \in [n]$.

$n = 8$:

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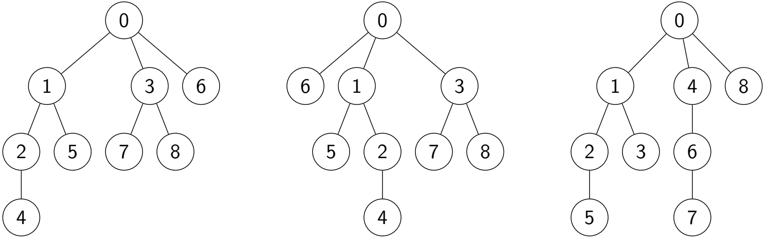

Subdiagonal functions

Definition

A **subdiagonal function** of domain $[n]$ is a function $f : [n] \to \{0\} \cup [n]$ such that $f(i) < i$, $\forall i \in [n]$.

$n = 8$:

n	1	2	3	4	5	6	7	8
$f(n)$	0	1	0	2	1	0	3	3

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Bijection between increasing trees and subdiagonal functions

- There exists a bijection between the set of increasing trees with $n+1$ vertices and the set of subdiagonal functions of domain $[n]$, which associates with each tree the function that sends each (non-root) vertex to its father.

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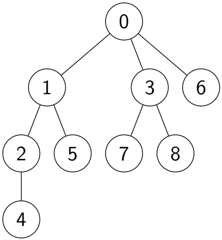

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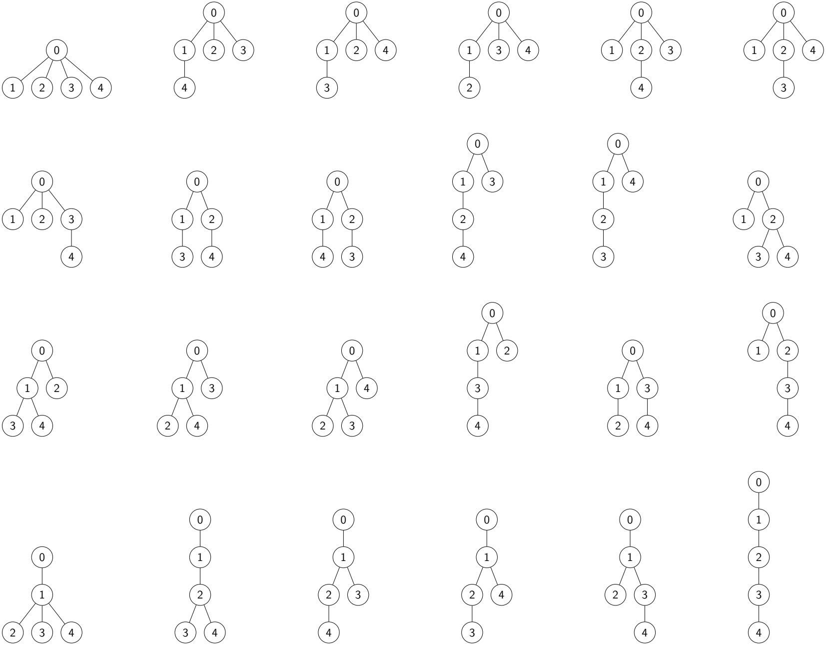

Proposition

Let $n \in \mathbb{N}_0$ and $\lambda \in \Gamma_n$, and let T_λ be the set of increasing trees with $n+1$ vertices such that, for each $i \in [n]$, vertex i has λ_{n-i} children. Then,

$$c_\lambda(1) = |T_\lambda|.$$

$\lambda = (0, 0, 1, 1)$
$(4, 3, 2, 1)$

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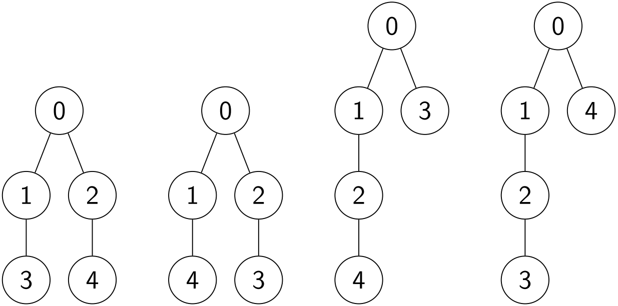

Proposition (Dual)

Let $n \in \mathbb{N}_0$ and $\lambda \in \Gamma_n$, and let F_λ be the set of the subdiagonal functions, f, of domain $[n]$, such that, for each $i \in [n]$, $\left|f^{-1}(\{i\})\right| = \lambda_{n-i}$. Then,

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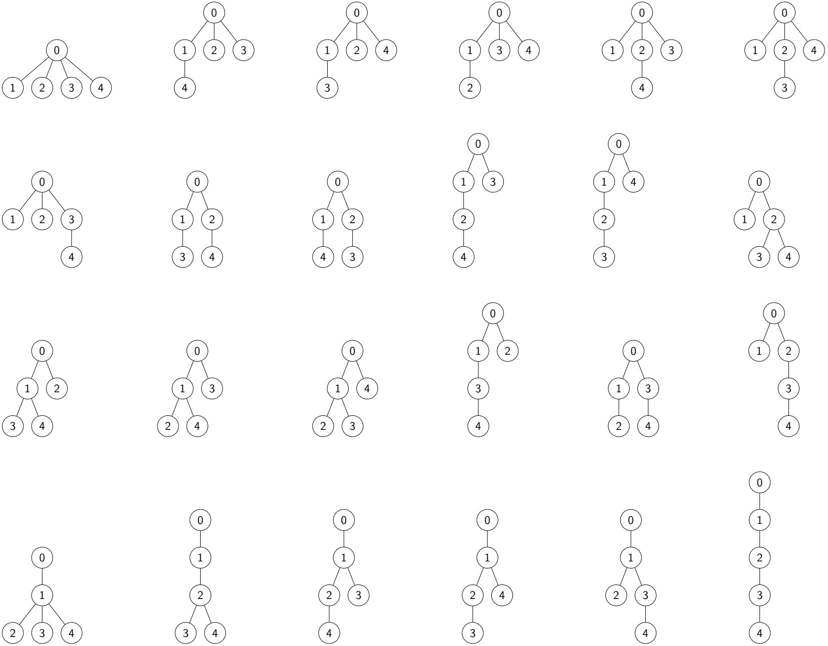

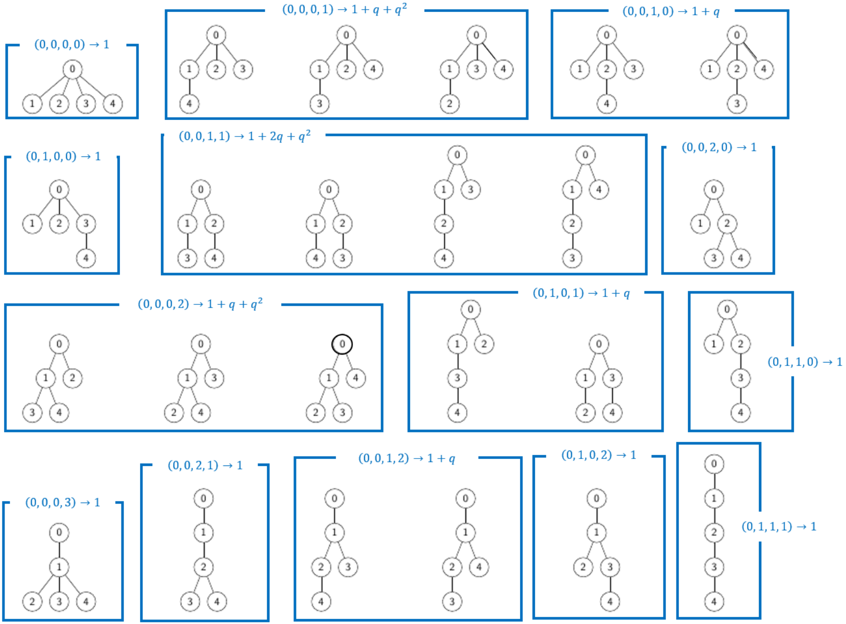
(0, 0, 0, 0) → 1
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(0, 1, 0, 0) → 1
(0, 0, 1, 1) → 1 + 2q + q^2
(0, 0, 2, 0) → 1
(0, 0, 0, 2) → 1 + q + q^2
(0, 1, 0, 1) → 1 + q
(0, 1, 1, 0) → 1
(0, 0, 0, 3) → 1
(0, 0, 2, 1) → 1
(0, 0, 1, 2) → 1 + q
(0, 1, 0, 2) → 1
(0, 1, 1, 1) → 1

Inversions

Definition

Given a function $f : [n] \to \{0\} \cup [n]$, an **inversion** is a pair $(a, b) \in [n] \times [n]$ such that $a < b$ and $f(a) > f(b)$.

Definition (Dual)

Given an increasing tree with $n+1$ vertices, an **inversion** is a pair $(a, b) \in [n] \times [n]$ such that $a < b$ and father$(a) >$ father(b).

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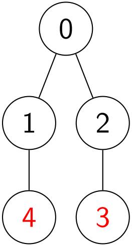

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Proposition

Let $n \in \mathbb{N}_0$, $\lambda \in \Gamma_n$ and $k \in \mathbb{N}_0$. Then, $c_\lambda[k]$ is the number of functions in F_λ with k inversions.

$\lambda = (0, 0, 1, 1)$

$(4\ 3\ 2\ 1)$

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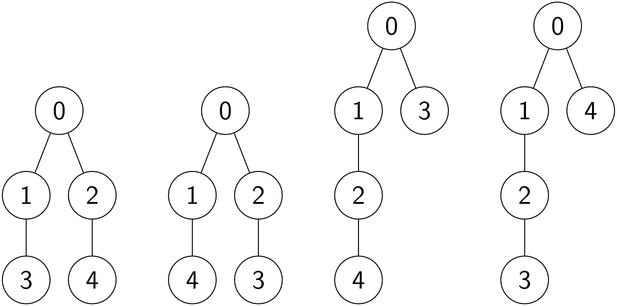

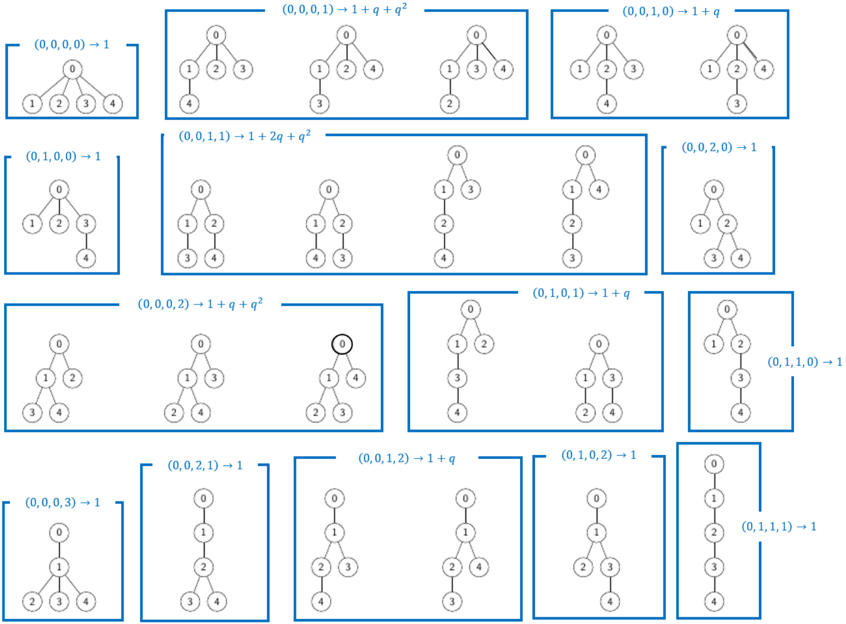
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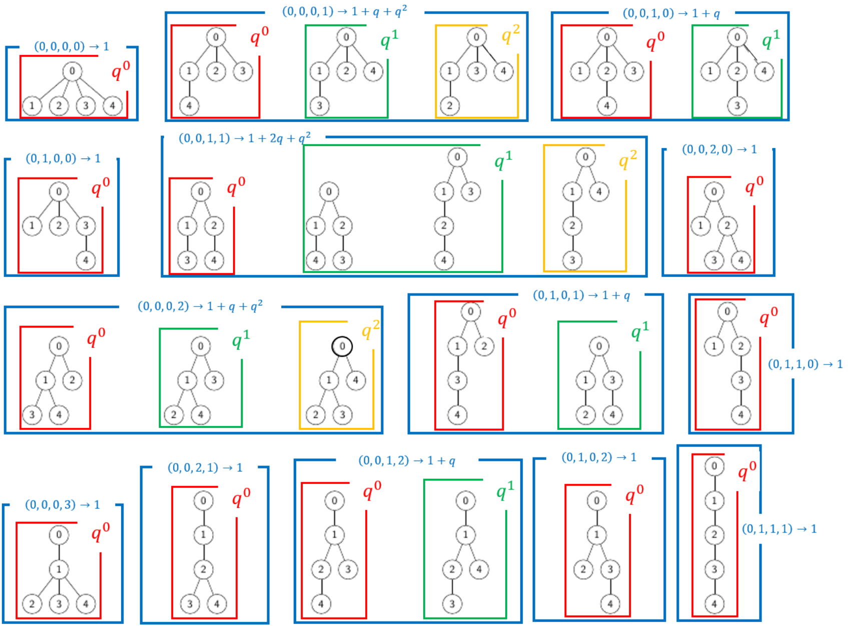
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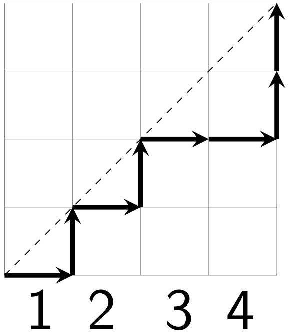
How many normal pseudo-compositions are there?



$\rightarrow$

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