Combinatorial Howe duality

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11th Combinatorics Days
Lisboa
21 January 2022

Howe duality in type A

Let $n, m \in \mathbb{Z}_{\geq 1}$ and consider the Lie algebras $\mathfrak{gl}_n, \mathfrak{gl}_m$ over $\mathbb{C}$.

$$\left\{ \begin{array}{c} \text{integral} \\ \text{dominant} \\ \text{weights for } \mathfrak{gl}_n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{partitions} \\ \text{of length} \\ \text{at most } n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Young diagrams} \\ \text{with at most} \\ n \text{ rows} \end{array} \right\}$$

- λ int. dominant $\mathfrak{gl}_n$-weight $\rightsquigarrow V_n(\lambda)$ irr. highest weight module,
- $\mathbb{C}^n \otimes \mathbb{C}^m$ natural $\mathfrak{gl}_n \times \mathfrak{gl}_m$-module $\rightsquigarrow \mathrm{Sym}(\mathbb{C}^n \otimes \mathbb{C}^m)$, $\Lambda(\mathbb{C}^n \otimes \mathbb{C}^m)$.

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Theorem (Howe duality)

We have the $\mathfrak{gl}_n \times \mathfrak{gl}_m$-module decompositions

$$\begin{aligned} \mathrm{Sym}(\mathbb{C}^n \otimes \mathbb{C}^m) &\simeq \sum_{\mathrm{len}(\lambda) \leq \min\{m,n\}} V_n(\lambda) \otimes V_m(\lambda), \\ \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m) &\simeq \sum_{\lambda \in [n \times m]} V_n(\lambda) \otimes V_m(\lambda'). \end{aligned}$$

Example

For $n = 3, m = 2$, consider the homogeneous component $\Lambda^3(\mathbb{C}^3 \otimes \mathbb{C}^2)$.

By the previous theorem, we have, as $\mathfrak{gl}_3 \times \mathfrak{gl}_2$-modules,

$$\Lambda^3(\mathbb{C}^3 \otimes \mathbb{C}^2) \simeq V_3(\ydiagram{2,1}) \otimes V_2(\ydiagram{2,1}) \oplus V_3(\ydiagram{1,1,1}) \otimes V_2(\ydiagram{3}).$$

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On the other hand, as $\mathfrak{gl}_3$-modules,

$$\begin{aligned}
\Lambda^3(\mathbb{C}^3 \otimes \mathbb{C}^2) &\simeq \Lambda^3(\mathbb{C}^3 \oplus \mathbb{C}^3) \\
&\simeq \Lambda^3\mathbb{C}^3 \oplus (\Lambda^2\mathbb{C}^3 \otimes \mathbb{C}^3) \oplus (\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3) \oplus \Lambda^3\mathbb{C}^3 \\
&\simeq V_3(\ydiagram{1,1,1}) \oplus (V_3(\ydiagram{1,1,1}) \oplus V_3(\ydiagram{2,1})) \oplus (V_3(\ydiagram{2,1}) \oplus V_3(\ydiagram{1,1,1})) \oplus V_3(\ydiagram{1,1,1}).
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&\simeq V_3(\ydiagram{1,1,1}) \oplus (V_3(\ydiagram{1,1,1}) \oplus V_3(\ydiagram{2,1})) \oplus (V_3(\ydiagram{2,1}) \oplus V_3(\ydiagram{1,1,1})) \oplus V_3(\ydiagram{1,1,1}).
\end{aligned}$$

We have

$$\begin{array}{ccll}
4 & = & \dim V_2(\ydiagram{3}) & \text{copies of } V_3(\ydiagram{1,1,1}) \\
2 & = & \dim V_2(\ydiagram{2,1}) & \text{copies of } V_3(\ydiagram{2,1})
\end{array}$$

This observation holds in general:

$$\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m) \simeq \bigoplus_{\substack{\nu \in \mathbb{Z}_{\geq 0}^m \\ |\nu| = N}} \underbrace{\Lambda^{\nu_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{\nu_m}\mathbb{C}^n}_{\Lambda^\nu \mathbb{C}^n}$$

and for $\nu \in [m \times n]$ and a partition $\lambda \in [n \times m]$, we have

$$[\Lambda^\nu \mathbb{C}^n : V_n(\lambda)] = \dim(V_m(\lambda'))_\nu. \qquad (1)$$

Goal: Give a bijective proof of (1).

For $\nu = (\nu_1, \dots, \nu_m) \in [m \times n]$, let

$$B_n^\nu = \{c_1 \otimes \cdots \otimes c_m \mid c_j \subseteq \{1, \dots, n\} \text{ s.t. } |c_j| = \nu_j\}.$$

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Theorem

We have a bijection

$$\begin{array}{ccc} \bigsqcup_{|\nu|=N} B_n^\nu & \xrightarrow{*} & \bigsqcup_{|\lambda|=N} B_m^\lambda \\ c_1 \otimes \cdots \otimes c_m & \longmapsto & d_1 \otimes \cdots \otimes d_n \end{array}$$

where

$$d_i = \{1 \leq j \leq m \mid i \in c_j\}.$$

If $b = c_1 \otimes \cdots \otimes c_m \in B_n^\nu$, each c_j is called (and represented by) a **column**.

Example Take $n = 4, m = 5$, $\nu = (3, 0, 2, 3, 2)$.

$$b = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \otimes \varnothing \otimes \begin{matrix} 1 \\ 4 \end{matrix} \otimes \begin{matrix} 1 \\ 2 \\ 4 \end{matrix} \otimes \begin{matrix} 1 \\ 3 \end{matrix} \longmapsto \begin{matrix} 1 \\ 3 \\ 4 \\ 5 \end{matrix} \otimes \begin{matrix} 1 \\ 4 \end{matrix} \otimes \begin{matrix} 1 \\ 5 \end{matrix} \otimes \begin{matrix} 3 \\ 4 \end{matrix} = b^*$$

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Observe that $b \in B_n^\nu \Leftrightarrow \mathrm{weight}(b^*) = \nu$. In our example,

- $b \in B_4^{(3,0,2,3,2)}$ and $\mathrm{weight}(b^*) = (\#1\text{'s in } b^*, \ldots, \#m\text{'s in } b^*) = (3,0,2,3,2)$,
- $b^* \in B_5^{(4,2,2,2)}$ and $\mathrm{weight}(b) = (4,2,2,2)$.

Definition

1. An element $b \in B_n^{\nu}$ is called **Yamanouchi** if each prefix of its reading has at least as many i as $i+1$ for all $1 \leq i < n$.
2. An element $a \in B_m^{\lambda}$ is called a **tableau** if removing the symbols "$\otimes$" yields a semistandard Young tableau.

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Theorem (Combinatorial Howe duality)

b is Yamanouchi if and only if b^* is a tableau.

Example

$$b = \begin{matrix} 1 \\ 2 \end{matrix} \otimes \begin{matrix} 1 \\ \end{matrix} \otimes \begin{matrix} 1 \\ 2 \end{matrix} \otimes \begin{matrix} 2 \\ 3 \end{matrix} \Leftrightarrow b^* = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \otimes \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} \otimes \begin{matrix} 4 \\ \\ \end{matrix}$$

- $\mathrm{read}(b) = 121132 \Rightarrow b$ is Yamanouchi.
- b^* yields the tableau $\begin{matrix} 1 & 1 & 4 \\ 2 & 3 & \\ 3 & 4 & \end{matrix}$.

Why does this prove (1)?

On the one hand, it is well-known that

$$\#\left\{a \in B_m^\lambda \text{ tableaux of weight } \nu\right\} = \dim(V(\lambda'))_\nu.$$

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On the other hand,

- B_n^ν is the **crystal** of the representation $\Lambda^\nu \mathbb{C}^n$,
- $b \in B_n^\nu$ is Yamanouchi iff b is a **highest weight vertex**.

Therefore, by general crystal theory,

$$\#\left\{b \in B_n^\nu \text{ Yamanouchi of weight } \lambda\right\} = [\Lambda^\nu \mathbb{C}^n : V(\lambda)].$$

The case of type C

Replacing $\mathfrak{gl}_n, \mathfrak{gl}_m$ by $\mathfrak{sp}_{2n}, \mathfrak{sp}_{2m}$, we have for $\nu \in [m \times 2n]$, $\lambda \in [n \times m]$,

$$[\Lambda^\nu \mathbb{C}^{2n} : V_n(\lambda)] = \dim(V_m(\overline{\lambda'}))_{\overline{\nu}}, \tag{2}$$

where $\overline{(\nu_1, \ldots, \nu_m)} = (n - \nu_m, \ldots, n - \nu_1)$.

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Plan of the proof

We use

1. an analogous crystal B_n^ν and a similar bijection $*$,
2. a combinatorial description of the highest weight vertices in B_n^ν, generalising the Yamanouchi property,
3. a notion of so-called **King tableaux** generalising usual tableaux and counting dimensions of the weight spaces.

We conclude by the same argument.

The bijection $*$ in type C

For $\nu = (\nu_1, \dots, \nu_m) \in [m \times 2n]$, let

$$B_n^\nu = \left\{ c_1 \otimes \cdots \otimes c_m \mid c_j \subseteq \{\overline{n} < \dots < \overline{1} < 1 < \dots < n\} \text{ s.t. } |c_j| = \nu_j \right\}.$$

Similarly, set for $\eta = (\eta_1, \dots, \eta_n) \in [n \times 2m]$,

$$\dot{B}_m^\eta = \left\{ d_1 \otimes \cdots \otimes d_n \mid d_i \subseteq \{1 < \overline{1} < \dots < n < \overline{n}\} \text{ s.t. } |d_i| = \eta_i \right\}.$$

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where $d_i = \{j \in \{1, \dots, m\} \mid i \in c_j\} \sqcup \{\overline{j} \in \{\overline{1}, \dots, \overline{m}\} \mid \overline{i} \notin c_j\}$.

Example

Take $n = 5, m = 2$, $\nu = (4, 6)$.

$$b = \begin{array}{c}\overline{3}\\ \overline{2}\\ 4\\ 5\end{array} \otimes \begin{array}{c}\overline{5}\\ \overline{2}\\ \overline{1}\\ 1\\ 2\\ 4\end{array} \quad \overset{*}{\longmapsto} \quad \begin{array}{c}1\\ \overline{2}\end{array} \otimes \overline{2} \otimes 2 \otimes \begin{array}{c}1\\ \overline{1}\\ 2\\ \overline{2}\end{array} \otimes \begin{array}{c}1\\ \overline{1}\end{array} = b^*$$

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Moreover, if we define

$$\begin{aligned} \mathrm{weight}(b) &= (\#\overline{n}\text{'s} - \#n\text{'s}, \dots, \#\overline{1}\text{'s} - \#1\text{'s}) \quad \text{and} \\ \mathrm{weight}(b^*) &= (\#m\text{'s} - \#\overline{m}\text{'s}, \dots, \#1\text{'s} - \#\overline{1}\text{'s}), \ \text{we have} \end{aligned}$$

$$b \in B_n^\nu \Leftrightarrow \mathrm{weight}(b^*) = \overline{\nu} \quad \text{and} \quad b^* \in B_m^\eta \Leftrightarrow \mathrm{weight}(b) = \overline{\eta}.$$

Example Above, we have $\nu = (4,6)$, so that $\overline{\nu} = (-1,1)$ and $\eta = (0,-2,1,1,0)$ so that $\overline{\eta} = (2,1,1,4,2)$.

King tableaux

Definition

A semistandard tableau on $\{1 < \overline{1} < \cdots < m < \overline{m}\}$ is called **King** if each entry in row j is at least equal to j.

Example The tableau
$$\begin{array}{cccc} 1 & \bar{1} & 2 & 2 \\ \bar{1} & 3 & \bar{3} & \\ 3 & & & \end{array}$$
is semistandard but not King.

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Moreover, an element b is a highest weight vertex in the crystal B_n^ν if and only if each prefix of its reading has partition weight.

Remark $\dot{B}_m^\eta$ is not a crystal!

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Remark $\dot{B}_m^{\eta}$ is not a crystal!

Theorem (G.-Guilhot-Lecouvey 2021)

b is a highest weight vertex iff b^* is a King tableau.

Example

Take $n = 4, m = 3$.

$$b = \begin{matrix}\bar{4}\\ \bar{3}\\ \end{matrix} \otimes \begin{matrix}\bar{2}\\ \bar{1}\\ 1\end{matrix} \otimes \begin{matrix}\bar{4}\\ \bar{3}\\ 3\end{matrix} \Leftrightarrow b^* = \begin{matrix}1\\ \bar{2}\\ 3\end{matrix} \otimes \begin{matrix}1\\ 3\\ \end{matrix} \otimes \begin{matrix}2\\ \bar{3}\\ \end{matrix} \otimes \begin{matrix}2\\ \\ \end{matrix}$$

- $\mathrm{read}(b) = \bar{4}\bar{3}\bar{2}\bar{1}1\bar{4}\bar{3}3 \Rightarrow b$ is a highest weight vertex,
- b^* is a King tableau: $\begin{matrix}1 & 1 & 2 & 2\\ \bar{2} & 3 & \bar{3} & \\ 3 & & & \end{matrix}$.

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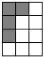
We have

$\nu = (2, 3, 3)$ so $\overline{\nu} = (1, 2, 2)$ and

$\lambda = (2, 1, 1, 0) =$  so $\overline{\lambda'} =$  $= (4, 3, 1)$.

Generalised Howe duality in type C

It is possible to prove a more general duality.

In (2), replace

$$\Lambda^\nu \mathbb{C}^{2n} = V_n(1^{\nu_1}) \otimes \cdots \otimes V_n(1^{\nu_m}) \quad \text{by}$$

$$V_n^{X_1}(\mu^{(1)}) \otimes \cdots \otimes V_n^{X_r}(\mu^{(r)})$$

where for all $1 \leq j \leq r$,

- $X_j \in \{A, C\}$,
- $\mu^{(j)}$ is a partition in $[n \times m_j]$ with $\sum m_j = m$,
- $V_n^{X_j}(\mu^{(j)}) =$ irr. rep. of type X_j with highest weight $\mu^{(j)}$.

For $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$, let $\widehat{\boldsymbol{\mu}} = (\widehat{\mu^{(r)}}, \dots, \widehat{\mu^{(1)}})$ where $\widehat{\alpha} = \overline{\alpha'}$.

Consider the subalgebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ of $\mathfrak{sp}_{2n}$ where

$$\mathfrak{g} = \begin{cases} \mathfrak{gl}_{m_j} & \text{if } X_j = A \\ \mathfrak{sp}_{2m_j} & \text{if } X_j = C. \end{cases}$$

$\widehat{\boldsymbol{\mu}}$ is a dominant weight for $\mathfrak{g} \rightsquigarrow$ irr. h.w. representation $V_m(\widehat{\boldsymbol{\mu}})$.

For $\lambda \in [n \times m]$, set

$$M_{\widehat{\boldsymbol{\mu}}}^{\widehat{\lambda}} = \text{ multiplicity of } V_m(\widehat{\boldsymbol{\mu}}) \text{ in } \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{sp}_{2m}} V_m(\widehat{\lambda}).$$

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Theorem (G.-Guilhot-Lecouvey 2021)

$$M_{\widehat{\boldsymbol{\mu}}}^{\widehat{\lambda}} = [V_n^{X_1}(\mu^{(1)}) \otimes \cdots \otimes V_n^{X_r}(\mu^{(r)}) : V_n(\lambda)].$$

Remark $r = m$, $m_j = 1$, $X_j = C \Rightarrow$ Identity (2).