Sharp lower bounds on the least eigenvalue of graphs determined from edge clique partitions

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Joint work with Domingos M. Cardoso and Rui Duarte.

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Content

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Definitions and notation

* Let's consider G as a simple graph (without multiple edges and loops).
* The set of vertices of G is denoted as $V(G)$.
* The set of edges of G is denoted as $E(G)$.
* A **complete graph** is a graph in which every pair of vertices is adjacent.
* A **clique** is a subset of $V(G)$ that induces a complete subgraph.
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Edge clique partition

An **edge clique partition** (ECP for short), introduced in [12, Orlin 1977], is a partition of $E(G)$ by cliques.

Figure: A graph G and an ECP of G, on the right, where the edges with the same color belong to the same part.

The **content of a graph** G, denoted by $C(G)$, was defined as the minimum number of edge disjoint cliques whose union includes all the edges of G.

Such minimum ECP is called in [12] **content decomposition** of G. As proved in [12], in general, the determination of $C(G)$ is NP-Complete.

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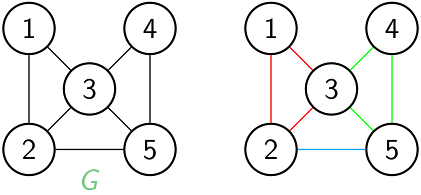

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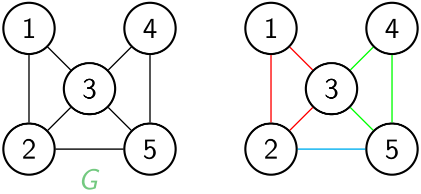

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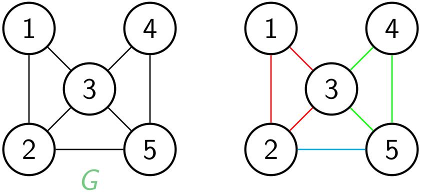

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Consider a graph G and an ECP, $P = \{E_i \mid i \in I\}$. Then $V_i = V(G[E_i])$ is a clique of G for every $i \in I$.

Clique degree

The **clique degree** of v relative to P, denoted $m_v(P)$, is the number of cliques V_i containing the vertex v.

$$m_v(P) = |\{i \in I \mid v \in V(G[E_i])\}|, \forall v \in V(G)$$

Maximum clique degree

The **maximum clique degree** of G relative to P, denoted $m_G(P)$, is the maximum of clique degrees of the vertices of G relative to P.

$$m_G(P) = \max\{m_v \mid v \in V(G)\}$$

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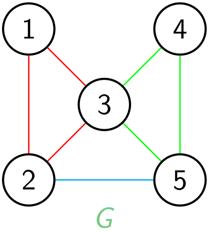

$P = \{\{12, 23, 31\}, \{34, 45, 53\}, \{25\}\}$ is a content decomposition of G.

$$m_v(P) = 2, \text{if } v \in \{2, 3, 5\}$$
$$m_v(P) = 1, \text{if } v \in \{1, 4\}.$$

Therefore, $m_G(P) = 2$.

It is clear that if P is an ECP of G, then $m_G(P) \leq |P|$.

In particular, if P is a content decomposition of G, then $m_G(P) \leq C(G)$.

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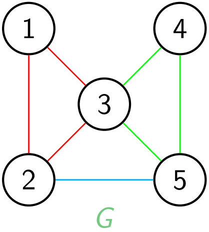

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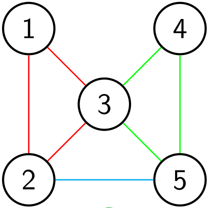

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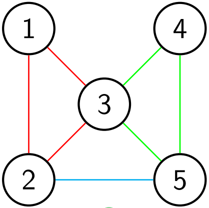

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For every integer $k \geq 2$, there exists a connected graph G_k that admits an ECP, P_k, such that $m_{G_k}(P_k) = k$.

Proof idea

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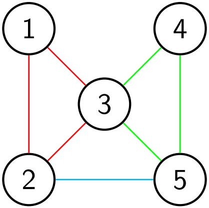

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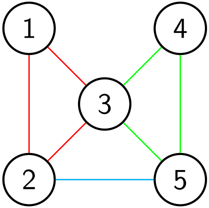

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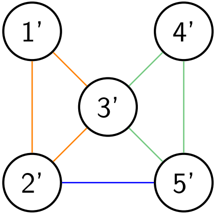

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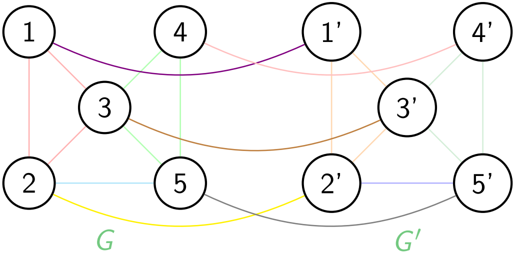

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Let $P = \{E_i \mid i \in I\}$ be an ECP of a graph G, $m = m_G(P)$ and $m_v = m_v(P)$ for every $v \in V(G)$. Then

1. *If μ is an eigenvalue of G, then $\mu \geq -m$.*
2. *$-m$ is an eigenvalue of G if and only if there exists a vector $X \neq 0$ such that*
   - (a) *$\sum_{j \in V(G[E_i])} x_j = 0$ for every $i \in I$ and*
   - (b) *$\forall v \in V(G), x_v = 0$ whenever $m_v \neq m$*

In the positive case, X is an eigenvector associated with the eigenvalue $-m$.

Since $m_G(P) = 2$, for every $\mu \in \sigma(G)$, $\mu \geq -2$.

$\sigma(G) = \{-1.473, -0.463, 0.118, 0.618, 2.935\}$

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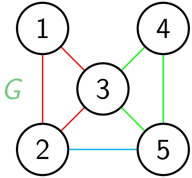

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The last theorem provides the spectral lower bound for the content of a graph which appears in [10, Hoffman 1972].

Corollary

Let μ be the least eigenvalue of a graph G. Then $-\mu \leq C(G)$.

Corollary

Let G be a graph of order n and let X be a vector of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then $X \in \mathcal{E}_G(-m)$ if and only if the conditions 2a and 2b of the previous Theorem hold.

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Let P be an ECP of a graph G. If $-m_G(P)$ is an eigenvalue of G, then for every ECP of G, P', $m_G(P') \geq m_G(P)$.

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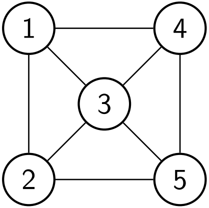

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ECP of G

Since $m_v(P) = 2$ for every $v \in V(G)$, $m_G(P) = 2$ and then every eigenvalue of G if greater or equal to -2.

The vector $X = [1, -1, 0, -1, 1]^T$ (on the right) fulls the necessary and sufficient conditions 2a and 2b of previous theorem and thus the least eigenvalue of G is equal to -2.

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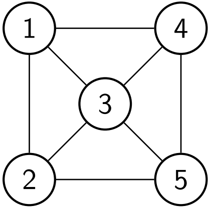

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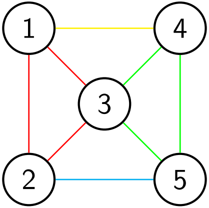

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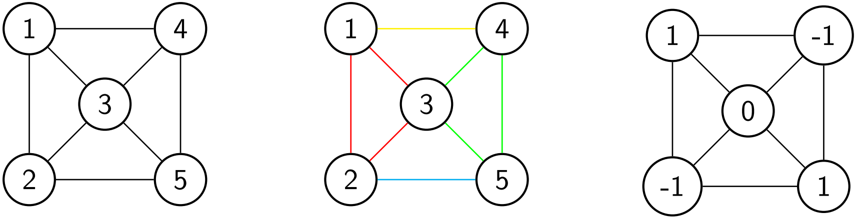

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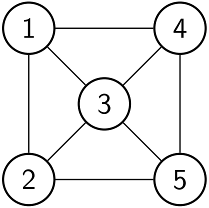

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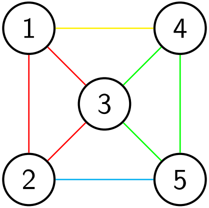

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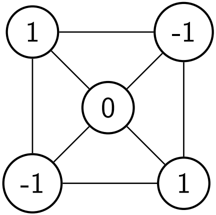

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Application to n-Queens' graph

$\mathcal{Q}(n)$

***n*-Queen's graph**, $\mathcal{Q}(n)$, associated to $n \times n$ chessboard has $n \times n$ vertices, corresponding to each square of the $n \times n$ chessboard.
Two vertices of $\mathcal{Q}(n)$ are adjacent if and only if they are in the same row or column or diagonal of the chessboard.

Figure: $\mathcal{Q}(3)$.

For $\mathcal{Q}(n)$, we will consider the ECP with maximal cliques in all the 4 edge directions, that is, each block of this partition is the clique defined by the edges whose vertices are in each row, each column and each diagonal of the chessboard.

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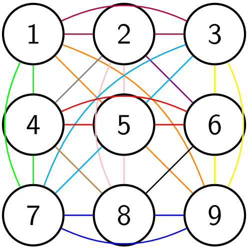

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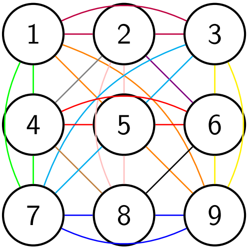

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Theorem

Let $n \in \mathbb{N}$ *such that* $n \geq 4$.

1. *If* μ *is an eigenvalue of* $\mathcal{Q}(n)$, *then* $\mu \geq -4$.
2. $-4 \in \sigma(\mathcal{Q}(n))$ *if and only if there exists a vector* $X \in \mathbb{R}^{n^2} \setminus \{\mathbf{0}\}$ *such that*
   1. $\sum_{j=1}^{n} x_{(k,j)} = 0$ *and* $\sum_{i=1}^{n} x_{(i,k)} = 0$, *for every* $k \in [n]$,
   2. $\sum_{i+j=k+2} x_{(i,j)} = 0$, *for every* $k \in [2n-3]$,
   3. $\sum_{i-j=k+1-n} x_{(i,j)} = 0$, *for every* $k \in [2n-3]$,
   4. $x_{(1,1)} = x_{(1,n)} = x_{(n,1)} = x_{(n,n)} = 0$.

   In the positive case, X *is an eigenvector associated with the eigenvalue* -4.

Application to n-Queens' graph

For an easier representation of the vectors, they are displayed over the chess-board.

We need to introduce the family of vectors

$$\mathcal{F}_n = \{X_n^{(a,b)} \in \mathbb{R}^{n^2} \mid (a,b) \in [n-3]^2\}$$

where $X_n^{(a,b)}$ is the vector defined by

$$\left[X_n^{(a,b)}\right]_{(i,j)} = \begin{cases} [X_4]_{(i-a+1, j-b+1)}, & \text{if } (i,j) \in A \times B; \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

with $A = \{a, a+1, a+2, a+3\}$, $B = \{b, b+1, b+2, b+3\}$ and

$$X_4 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & -1 & 0 \\ \hline -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 \\ \hline 0 & -1 & 1 & 0 \\ \hline \end{array}.$$

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1	0	0	-1
0	-1	1	0

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For instance, for $n = 5$, $\mathcal{F}_5$ is the family of four vectors depicted in the next figure.

0	1	-1	0	0
-1	0	0	1	0
1	0	0	-1	0
0	-1	1	0	0
0	0	0	0	0

0	0	1	-1	0
0	-1	0	0	1
0	1	0	0	-1
0	0	-1	1	0
0	0	0	0	0

0	0	0	0	0
0	1	-1	0	0
-1	0	0	1	0
1	0	0	-1	0
0	-1	1	0	0

0	0	0	0	0
0	0	1	-1	0
0	-1	0	0	1
0	1	0	0	-1
0	0	-1	1	0

Figure: The vectors $X_5^{(1,1)}$, $X_5^{(1,2)}$, $X_5^{(2,1)}$, and $X_5^{(2,2)}$.

Theorem

For $n \geq 4$, -4 is an eigenvalue of $\mathcal{Q}(n)$ with multiplicity $(n-3)^2$ and $\mathcal{F}_n$ is a basis for $\mathcal{E}_{\mathcal{Q}(n)}(-4)$.

Application to n-Queens' graph

For instance, for $n = 5$, $\mathcal{F}_5$ is the family of four vectors depicted in the next figure.

0	1	-1	0	0
-1	0	0	1	0
1	0	0	-1	0
0	-1	1	0	0
0	0	0	0	0

0	0	1	-1	0
0	-1	0	0	1
0	1	0	0	-1
0	0	-1	1	0
0	0	0	0	0

0	0	0	0	0
0	1	-1	0	0
-1	0	0	1	0
1	0	0	-1	0
0	-1	1	0	0

0	0	0	0	0
0	0	1	-1	0
0	-1	0	0	1
0	1	0	0	-1
0	0	-1	1	0

Figure: The vectors $X_5^{(1,1)}$, $X_5^{(1,2)}$, $X_5^{(2,1)}$, and $X_5^{(2,2)}$.

Theorem

For $n \geq 4$, -4 is an eigenvalue of $\mathcal{Q}(n)$ with multiplicity $(n-3)^2$ and $\mathcal{F}_n$ is a basis for $\mathcal{E}_{\mathcal{Q}(n)}(-4)$.

Content

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