A walk through down and up operators in search of deformation

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- But what are differential posets?
- Differential posets are a familly of structures introdused by Richard Stanley in 1988 in a paper with the title "Differential posets".

Up and down operators

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- Given any poset P, we may define an abstract vector space $\mathbb{F}P = \oplus_{x \in P} \mathbb{F}x$ of finite linear combinations of elements of P with coefficients in $\mathbb{F}$.
- If in addition P is locally finite and each element of P is a member of only finitely many cover relations, we may define two linear transformations d and u on $\mathbb{F}P$ as follows:

$$dx = \sum_{y \lessdot x} y \qquad \text{and} \qquad ux = \sum_{x \lessdot y} y \tag{0.1}$$

where $\lessdot$ stands for the cover relationship and we extend both to all of $\mathbb{F}P$ by linearity.

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where $\lessdot$ stands for the cover relationship and we extend both to all of $\mathbb{F}P$ by linearity.
- The operators u and d in (0.1) will be called up and down operators which keep track of all possible steps "up" and "down" in the Hasse diagram from x and have been introduced as Schur operators by Fomin.

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$$dux = \sum_{\substack{y,z \\ z \lessdot y \text{ and } x \lessdot y}} z.$$

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  An element $z \in P$ appears in this sum exactly k times, where k is the number of elements of P which are covered by both x and z.
- Thus, we see that $(du - ud)x = x$ if and only if x is covered by exactly one more element than it covers and for each $z \neq x \in P$, the number of elements covering both x and z is equal to the number of elements covered by both x and z.

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Proposition
For a differential poset P, we have $du^n = nu^{n-1} + u^n d$ for all $n \geq 1$.

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- Stanley found that many of the interesting enumerative and structural properties of Young's lattice could be deduced from the relation $du - ud = I$, for I the identity transformation on $\mathbb{F}P$.
- Stanley also considered more general posets which satisfy the relation $du - ud = rI$ for some fixed positive integer r and he referred to these kind of posets as "r-differential".

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- In his study of uniform posets, Paul Terwilliger considered finite ranked posets P whose down and up operators satisfy the following relation

$$d_i d_{i+1} u_i = \alpha_i d_i u_{i-1} d_i + \beta_i u_{i-2} d_{i-1} d_i + \gamma_i d_i, \tag{0.4}$$

  where d_i and u_i denote the restriction of d and u to the elements of rank i.

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- There is an analogous second relation,

$$d_{i+1} u_i u_{i-1} = \alpha_i u_{i-1} d_i u_{i-1} + \beta_i u_{i-1} u_{i-2} d_{i-1} + \gamma_i u_{i-1}, \tag{0.5}$$

which holds automatically because d_{i+1} and u_i are adjoint operators relative to a certain bilinear form.

(q, r)-differential posets

- In many classical cases the constants in the above relations (relations (0.4) and (0.5)) do not depend on the rank of the poset. A particular instance of this provides a q-analogue of the notion of differential poset.

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Definition
A partially ordered set whose down and up operators satisfy

$$\begin{aligned} d^2u &= q(q+1)dud - q^3ud^2 + rd \\ du^2 &= q(q+1)udu - q^3u^2d + ru, \end{aligned} \tag{0.7}$$

where q and r are fixed complex numbers is said to be "(q, r)-differential poset".

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- Examples of (q, r)-differential posets include the posets of alternating forms, quadratic forms, or Hermitian forms over a finite field.

Down-up algebras

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Definition
We say a unital associative algebra $A = A(\alpha, \beta, \gamma)$ over the complex numbers $\mathbb{C}$ with generators u, d and defining relations

$$\begin{aligned} d^2u &= \alpha dud - \beta ud^2 + \gamma d \\ du^2 &= \alpha udu - \beta u^2d + \gamma u, \end{aligned} \tag{0.9}$$

where α, β, γ are fixed but arbitrary elements of $\mathbb{C}$, is a down-up algebra.

Generalized down-up algebras

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Definition
Let $f \in \mathbb{F}[h]$. Define $L := L(f, r, s, \gamma)$ to be the unital associative $\mathbb{F}$-algebra generated by x, y and h with relations

$$yh - rhy = \gamma y, \quad hx - rxh = \gamma x, \quad yx - sxy + f(h) = 0;$$

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- Over an algebraically closed field, generalized down-up algebras include all down-up algebras.

Quantum generalized Heisenberg algebras

Working over an arbitrary field $\mathbb{F}$ (unless otherwise stated), we introduce a generalization of the class of generalized Heisenberg algebras by deforming and generalizing the relation $yx - xy = f(h) - h$, turning it into a skew-commutation relation and allowing the skew-commutator to equal a generic polynomial, independent of f.

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Definition
For fixed and independent $f, g \in \mathbb{F}[h]$ and $q \in \mathbb{F}$, we define $\mathcal{H}_q(f, g)$ as the unital associative algebra over $\mathbb{F}$ generated by x, y, and h subject to the following relations

$$yh = f(h)y, \quad hx = xf(h), \quad yx = qxy + g(h). \tag{0.12}$$

Quantum generalized Heisenberg algebras

- Our main motivation for introducing a generalization of the class of *GHA*s, besides providing a broader framework for the investigation of the underlying physical systems, comes from the observation that the classes of generalized Heisenberg algebras and of (generalized) down-up algebras intersect, although neither one contains the other.

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Proposition
The GHA $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra (GDUA) if and only if $\deg f \leq 1$.

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Proposition
The GHA $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra (GDUA) if and only if $\deg f \leq 1$.

- But not all (generalized) down-up algebras are *GHA*s. And it became an incentive to develop a generalization of the concept of *GHA* to a new class which includes all *GHA*s and all *GDUA*s.

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- The universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ is the quotient of the free algebra $\mathbb{F}\langle X, Y, H\rangle$ modulo the two sided ideal I generated by elements $XY - YX - H$, $XH - HX$ and $YH - HY$. It is easy to see that $U(\mathfrak{h}_1)$ is isomorphic to the quantum generalized Heisenberg algebra$\mathcal{H}_1(h, -h)$ if we consider the correspondence $X \leftrightarrow x$, $Y \leftrightarrow y$ and $H \leftrightarrow h$.

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- Consider the 3-dimensional Lie algebra $\mathfrak{sl}_2$, with basis elements x, y, h and Lie bracket given by $[x, h] = 2x$, $[h, y] = 2y$ and $[y, x] = h$. We can view its enveloping algebra as *qGHA* $\mathcal{H}_1(h-2, h)$.

Quantum generalized Heisenberg algebras

- Fix $g \in \mathbb{C}[x]$. Define $S = \mathbb{C}[A, B, H]$ subject to the relations

  $$[H, A] = A, \qquad [H, B] = -B, \qquad AB - BA = g(H).$$

  Then S will be called the Smith algebra due to Paul Smith, and it is easy to see that S is isomorphic to the quantum generalized Heisenberg algebra $\mathcal{H}_1(h-1, g)$ if we consider the correspondence $B \leftrightarrow x$, $A \leftrightarrow y$ and $H \leftrightarrow h$.

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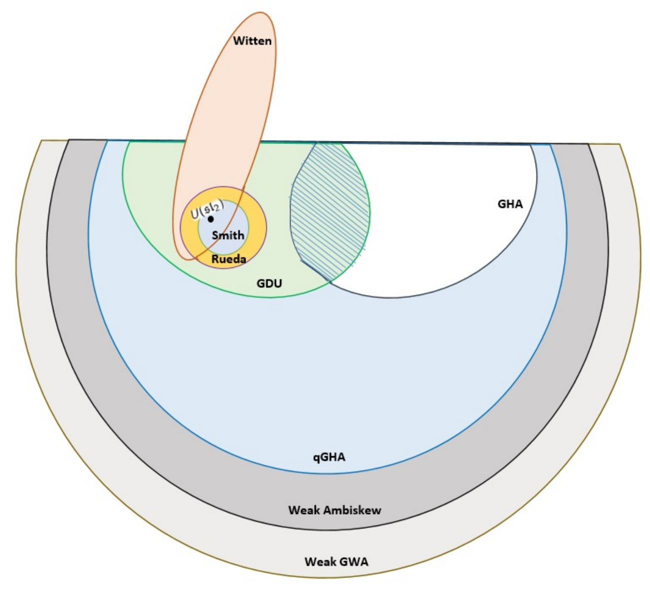
- Fix $g \in \mathbb{C}[x]$ and $ç \in \mathbb{C}, ç \neq 0$. Define $R = \mathbb{C}[A, B, H]$ subject to the relations

$$[H, A] = A, \qquad [H, B] = -B, \qquad AB - çBA = g(H).$$

  Then R will be called the Rueda algebra due to Sonia Rueda, and it is easy to see that R is isomorphic to the quantum generalized Heisenberg algebra $\mathcal{H}_ç(h-1, g)$ if we consider the correspondence $B \leftrightarrow x$, $A \leftrightarrow y$ and $H \leftrightarrow h$.

Graphical view of subclasses



Seeing $\mathcal{H}_q(f, g)$ as a weak ambiskew polynomial ring

Proposition
For $q \in \mathbb{F}$, $f, g \in \mathbb{F}[h]$, we construct $\mathcal{H}_q(f, g)$ as the ambiskew polynomial ring $R(\mathbb{F}[h], \sigma, g(h), q)$, where σ is the endomorphism of $\mathbb{F}[h]$ defined by $\sigma(h) = f(h)$.

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- And we have Poincare Birkhoff Witt type basis $\{x^i h^j y^k \mid i, j, k \in \mathbb{Z}_{\geq 0}\}$ of $\mathcal{H}_q(f, g)$.

Seeing $\mathcal{H}_q(f, g)$ as a weak ambiskew polynomial ring

- Viewing $\mathcal{H}_q(f, g)$ as a 2-step Ore extension (or as an ambiskew polynomial ring) also helps to determine when $\mathcal{H}_q(f, g)$ is a domain.

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Lemma
$\mathcal{H}_q(f, g)$ is a domain if and only if $q \neq 0$ and $\deg f \geq 1$.

Seeing $\mathcal{H}_q(f,g)$ as a weak generalized Weyl algebras

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Proposition
For $q \in \mathbb{F}$, and $f,g \in \mathbb{F}[h]$, $\mathcal{H}_q(f,g)$ is isomorphic to the wGWA $A(\sigma,\omega)$ for $A = \mathbb{F}[h,\omega]$ and σ the endomorphism of A defined by $\sigma(h) = f(h)$ and $\sigma(\omega) = q\omega + g(h)$.

Relation between GDUAs and qGHAs

Corollary
$\mathcal{H}_q(f,g)$ *is isomorphic to a GDUA if and only if* $\deg f \leq 1$.

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Proposition
The algebra $\mathcal{H}_q(f, g)$ is right (resp. left) Noetherian if and only if $\deg f = 1$ *and* $q \neq 0$.

Weight $\mathcal{H}_q(f,g)$-modules

Now set

$$\mathsf{S}_f = \{\lambda : \mathbb{Z} \longrightarrow \mathbb{F} \mid f(\lambda(i)) = \lambda(i+1), \text{ for all } i \in \mathbb{Z}\} \tag{0.19}$$

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and for given $\lambda \in \mathsf{S}_f$, we define

$$\mathsf{T}_{q,g,\lambda} = \{\mu : \mathbb{Z} \longrightarrow \mathbb{F} \mid \mu(i+1) = q\mu(i) + g(\lambda(i)),\ \text{for all } i \in \mathbb{Z}\}\,.$$

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- Next we will construct universal weight modules:

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Definition
Given $\lambda \in \mathsf{S}_f$ and $\mu \in \mathsf{T}_{q,g,\lambda}$ we define the $\mathcal{H}_q(f,g)$-module $\mathsf{A}_{q,f,g}(\lambda,\mu)$ by setting $\mathsf{A}_{q,f,g}(\lambda,\mu) = \mathbb{F}[t^{\pm 1}]$, as vector spaces, with action given by

$$ht^i = \lambda(i)t^i, \quad xt^i = t^{i+1}, \quad yt^i = \mu(i)t^{i-1}, \quad \text{for all } i \in \mathbb{Z}. \qquad (0.21)$$

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Definition
We can identify $\mathsf{B}_{q,f,g}(\lambda,\mu)$ with $\mathbb{F}[t^{\pm 1}]$, so that the action will be given by:

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Weight $\mathcal{H}_q(f,g)$-modules

Now fix $\alpha \in \mathbb{F}$ and define $\nu_\alpha : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{F}$

$$\nu_\alpha(i) = \sum_{j=0}^{i-1} q^j g(f^{(i-1-j)}(\alpha)), \quad \text{for all } i \geq 0. \tag{0.24}$$

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adopting the convention that $yt^0 = 0$.

- It is a routine and easy work to check that these structures indeed define $\mathcal{H}_q(f,g)$-modules.

Simple $\mathcal{H}_q(f, g)$-modules

Next we will construct finite-dimensional simple $\mathcal{H}_q(f, g)$-modules as quotients of the modules $\mathsf{A}_{q,f,g}(\lambda, \mu)$, $\mathsf{B}_{q,f,g}(\lambda, \mu)$ and $\mathsf{C}_{q,f,g}(\alpha)$.

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Lemma

Let $\lambda \in S_f$, $\mu \in T_{q,g,\lambda}$ such that $|\lambda| = m \geq 1$ and assume there exists $k \geq 1$ such that $\mu(km) = \mu(0)$. Let $\ell = \min\{k \geq 1 : \mu(km) = \mu(0)\}$. Then for any $\gamma \in \mathbb{F}^*$

$$\mathsf{A}_{q,f,g}(\lambda,\mu)/\mathbb{F}[t^{\pm 1}](t^{\ell m} - \gamma) \text{ and } \mathsf{B}_{q,f,g}(\lambda,\mu)/\mathbb{F}[t^{\pm 1}](t^{|\lambda||\mu|} - \gamma)$$

are simple.

Simple $\mathcal{H}_q(f, g)$-modules

Lemma

Let $\alpha \in \mathbb{F}$ and ν_α be given as before. Suppose that $\nu_\alpha(n) = 0$, for some $n \geq 1$. Then $\mathbb{F}[t]t^n$ is a submodule of $\mathsf{C}_{q,f,g}(\alpha)$ and the quotient module $\mathsf{C}_{q,f,g}(\alpha)/\mathbb{F}[t]t^n$ is simple if and only if $\nu_\alpha(1)\cdots\nu_\alpha(n-1) \neq 0$.

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Proposition

Let V be a simple $\mathcal{H}_q(f,g)$-module with $\dim V = n$ such that $x^n V \neq 0$. Then there are $m, \ell \geq 1$, $\gamma \in \mathbb{F}^*$, $\lambda \in S_f$ and $\mu \in T_{q,g,\lambda}$ such that

$$V \simeq \mathsf{A}_{q,f,g}(\lambda, \mu)/\mathbb{F}[t^{\pm 1}](t^{\ell m} - \gamma)$$

and $n = \ell m$, $\lambda(0) = \alpha$, $|\lambda| = m$, $f^{(m)}(\alpha) = \alpha$ and $\mu(\ell m) = \mu(0)$.

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- The analogous result for n-dimensional simple modules V such that $y^n V \neq 0$ uses the dual modules $\mathrm{B}_{q,f,g}(\lambda, \mu)$.

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Proposition

Assume $\mathbb{F}=\overline{\mathbb{F}}$. Let V be a simple $\mathcal{H}_q(f,g)$-module with $\dim_{\mathbb{F}} V = n$ and $x^n V = 0 = y^n V$. Then there is $\alpha \in \mathbb{F}$ such that $\nu_\alpha(n)=0$ and $V \simeq \mathsf{C}_{q,f,g}(\alpha)/\mathbb{F}[t]t^n$.

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(3) $\mathsf{C}_{q,f,g}(\alpha)/\mathbb{F}[t]t^n$, *for some* $\alpha \in \mathbb{F}$ *such that* $\nu_\alpha(n) = 0$ *and* $\nu_\alpha(i) \neq 0$ *for all* $1 \leq i \leq n-1$.

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- Considering *qGHA*s defined over more general rings than $\mathbb{F}[h]$.

Thank you!

I would be happy to answer your questions.