Efficient Vectors for Simple Perturbed Consistent Matrices

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Introduction

Analytical Hierarchy Process (AHP) has become a useful tool for analysing decisions. This process, developed by Thomas L. Saaty in the 1970's.

It is used in a decision process with a finite set of alternatives, $X = \{x_1, \ldots, x_n\}$ and where the decision maker is expected to select the best option.

Introduction

Formally, the different alternatives are two by two compared using a given criteria. This process gives rise to a matrix $A = [a_{i,j}]$, where $a_{i,j} > 0$ expresses the degree of preference of the alternative i concerning to j. So, A satisfies

$$a_{i,j} = \frac{1}{a_{j,i}}. \tag{1}$$

Introduction

An $n \times n$ positive matrix $A = [a_{i,j}]$ is called a pairwise comparison matrix (briefly, a PC matrix) if it satisfies (1).

Throughout, we denote by $\mathcal{PC}_n$ the set of all $n \times n$ pairwise comparison matrices.

Introduction

A matrix $A = [a_{ij}] \in \mathcal{PC}_n$ is said to be transitive or consistent if

$$a_{i,j}a_{j,k} = a_{i,k},$$

for all $i, j, k = 1, \ldots, n$.

Introduction

It is known that a matrix A is transitive if and only if there exists a positive vector $w = [w_1, \dots, w_n]^T$ such that

$$A = ww^{-1},$$

where $w^{-1} = [w_1^{-1}, \dots, w_n^{-1}]$.

Introduction

$$A = \begin{bmatrix} 1 & \frac{x_1}{x_2} & \frac{x_1}{x_3} & \dots & \frac{x_1}{x_{n-1}} & \frac{x_1}{x_n} \\ \frac{x_2}{x_1} & 1 & \frac{x_2}{x_3} & \dots & \frac{x_2}{x_{n-1}} & \frac{x_2}{x_n} \\ \frac{x_3}{x_1} & \frac{x_3}{x_2} & 1 & \dots & \frac{x_3}{x_{n-1}} & \frac{x_3}{x_n} \\ \frac{x_4}{x_1} & \frac{x_4}{x_2} & \frac{x_4}{x_3} & \dots & \frac{x_4}{x_{n-1}} & \frac{x_4}{x_n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \frac{x_n}{x_1} & \frac{x_n}{x_2} & \frac{x_n}{x_3} & \dots & \frac{x_n}{x_{n-1}} & 1 \end{bmatrix},$$

where $x_1, \dots, x_n$ be arbitrary positive numbers. Any matrix in $\mathcal{PC}_2$ is consistent, but this is not generally true in $\mathcal{PC}_n$ with $n > 2$.

Introduction

When applying the AHP method, the PC matrices $A = [a_{ij}] \in \mathcal{PC}_n$ obtained in practice should be approximated by a consistent one, that is, we should found a positive vector

$$w = [w_1 \dots w_n]^T,$$

such that the ratios $\frac{w_i}{w_j}$ are as close as possible to the a_{ij}'s for all $i, j = 1, 2, \dots, n$.

Efficient vector

Definition

A positive vector

$$w = [w_1 \dots w_n],$$

is said to be efficient for $A = [a_{i,j}] \in \mathcal{PC}_n$ if there is no other vector $w' = [w'_1 \dots w'_n]$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right|, \qquad \text{for all } 1 \leq i, j \leq n, \tag{2}$$

with the inequality strict for at least one pair (i, j)

Efficient vectors

Definition

A simple perturbed consistent matrix is a PC-matrix that differs from a consistent matrix in just one entry above the main diagonal and its reciprocal.

Simple perturbed consistent matrix

Example

The matrix

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ \frac{1}{5} & 1 & 1 \end{bmatrix} \in \mathcal{PC}_3,$$

is a simple perturbed consistent matrix.

Principal vector

From Perron-Frobenius Theorem for positive matrices, we know that if A is a positive matrix, then there is an eigenvalue r of A such that

$$|\lambda| < r,$$

for any other eigenvalue λ of A different from r.

Principal vector

- r is a simple eigenvalue;
- Any associated eigenvector has nonzero entries with a constant sign.

Principal vector

- We call the eigenvalue r the *Perron eigenvalue* of A;
- The associated right eigenvector with the last entry equal to 1 is the principal vector of A.

Efficiency of Principal vector

In 2016, Nagy and Bozoky proved that the principal vector of a simple perturbed consistent matrix is always efficient.

In 2018, Nagy, Bozoky and Rebak proved that the principal vector of a double perturbed consistent matrix is always efficient.

Efficiency of Principal vector

However, in 2022, Fernandes and Furtado proved that the principal vector of a triple perturbed consistent matrix is not always efficient.

The digraph $G_{A,w}$

Given $A \in \mathcal{PC}_n$ and a vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$, we denote by $G_{A,w} = (V, E)$ the directed graph (digraph) whose vertex set is $V = \{1, \dots, n\}$ and arc set is

$$E = \{i \to j : \frac{w_i}{w_j} \geq a_{ij}, i \neq j\}. \tag{3}$$

The digraph $G_{A,w}$

Theorem(Blanquero, Carrizosa, Conde, 2006)

Let $A \in \mathcal{PC}_n$. A vector w is efficient for A if and only if $G_{A,w}$ is a strongly connected digraph; for all pairs of vertices i, j, with $i \neq j$, there is a directed path from i to j and from j to i in $G_{A,w}$.

Lemma

Let $A, B \in \mathcal{PC}_n$. *Suppose that* $B = QAQ^{-1}$, *where* Q *is a product of a permutation matrix and a diagonal matrix with positive diagonal entries. Then, the* $n \times 1$ *positive vector* w_A *is efficient for* A *if and only if* $w_B = Qw_A$ *is efficient for* B.

Simple perturbed consistent matrices

$$Z_n(\delta) := \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & \delta \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ \frac{1}{\delta} & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \end{bmatrix} \in \mathcal{PC}_n, \tag{4}$$

for some $\delta > 0$.

Lemma

Let $w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}^T$ *a positive vector.*

1. *If* $w_p = w_q$, *for some* $p, q \in \{2, \ldots, n-1\}$ *with* $p \neq q$, *then* w *is efficient for* $Z_n(\delta)$ *if and only if* $w(\{p\})$ *is efficient for* $Z_{n-1}(\delta)$.
2. *Let* σ *be a permutation of* $\{2, \ldots, n-1\}$. *Then,* w *is efficient for* $Z_n(\delta)$ *if and only if*

$$w' = \begin{bmatrix} w_1 & w_{\sigma(2)} & \cdots & w_{\sigma(n-1)} & w_n \end{bmatrix}^T \tag{5}$$

is efficient for $Z_n(\delta)$.

Theorem

Let $n > 3$*,* $\delta > 1$ *and* $w = \begin{bmatrix} w_1 & \cdots & w_{n-1} & 1 \end{bmatrix}^T$ *be a positive vector. Then* w *is efficient for* $Z_n(\delta)$ *if and only if*

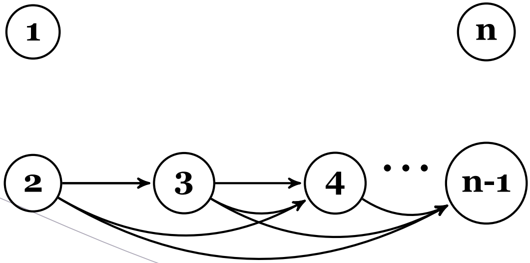
$$\delta \geq x_1 \geq x_i \geq 1, \textit{ for } i = 2, \ldots, n-1.$$

Proof: Let $A = [a_{ij}] = Z_n(\delta)$, and assume that $w = [w_1 \ \dots \ w_n], \ w_n = 1$, is efficient for A. Taking into account Lemma 5, we may assume that

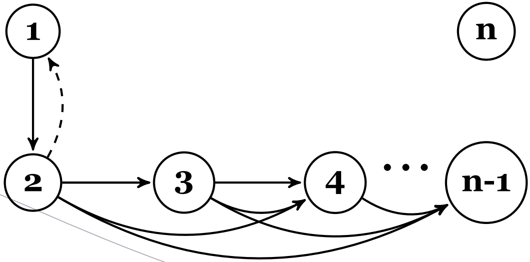
$$w_2 > \cdots > w_{n-1}. \tag{6}$$

If $i, j \in \{2, \dots, n-1\}$, $i < j$, since $w_i > w_j$ we have

$$\frac{w_i}{w_j} > 1 = a_{i,j}.$$


1
n
2
3
4
...
n-1

CASE 1: Assume that $w_1 \geq w_2$.



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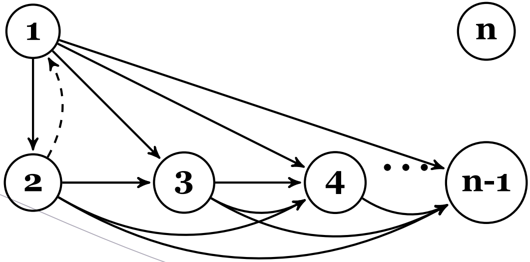

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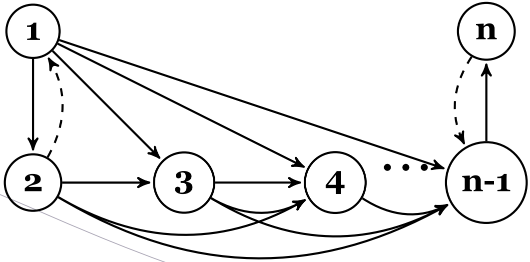

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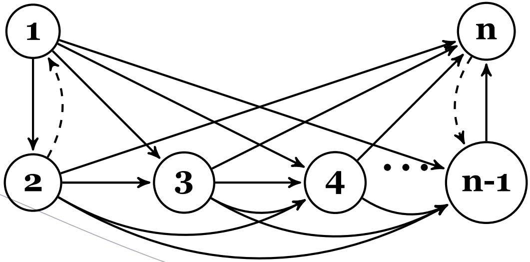

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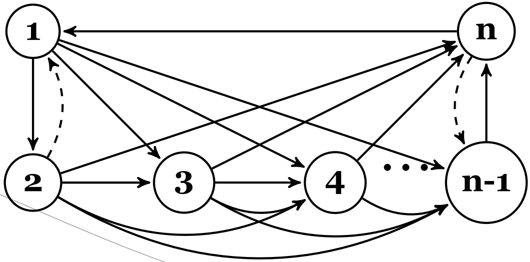

⋆ da Cruz, H.F., Fernandes, R., Furtado, S., Efficient vectors for simple perturbed consistent matrices, *International Journal of Approximate Reasoning*, 139, (2021), 54-68