# Representability of Cubes and of qualitive probability orders

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# Basic terminology of matroids

A **matroid** over a set $E$, $M = M(E)$ can be defined by its:

**list of Hyperplanes** -maximal flats distinct from $E$
**list of Cocircuit** - the sets complementary to a hyperplane.
**list of circuits** - minimal dependent subsets.

An **orientation** $\mathcal{M}(\mathcal{E})$ **of a matroid** $M = M(E)$ is:

list of **signed cocircuits** - partitions of the cocircuits of $M$ mimmiquing separation $\mathbb{R}^n$ by real hyperplanes
list of **signed circuits** - partitions of the circuits of $M$ mimiquing properties of $\mathbb{R}^n$ of convex partitions of minimal dependent subsets.

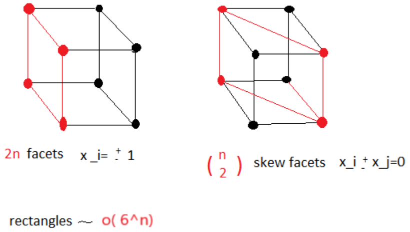
# Combinatorial Cubes

$[n] = \{1, \ldots, n\}$
$\alpha \subseteq [n] \equiv \alpha \in \{0,1\}^n = C^n.$

*A* **Combinatorial Cube** *is* **a matroid,** $M = M(C^n)$, *satisfying two conditions:*

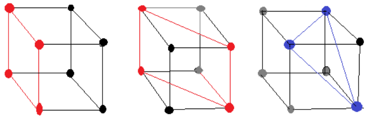
*(i) Every* **rectangle** *of* $C^n$ *is a* **circuit** *of* $M$.

*(ii) Every* **facet** *and* **skew-facet** *of* $C^n$ *is a* **hyperplane and a cocircuit** *of* $M$.

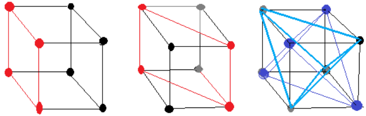


# Examples of cubes

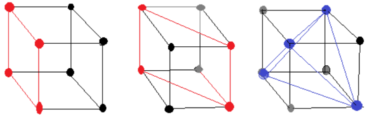
Real affine 3-cube - $Aff_{\mathbb{R}}(C^3)$



affine 3-cube over $Z_2$ - $Aff_{Z_2}(C^3)$



3-cube NON- representable over any field



# Orientability of Cubes I - IS 08,09, E.Gioan/IS 16

**General procedure to generate cubes from giving ones** - EXCEPT for $Aff_{\mathbb{R}}(C^n)$ , all known ones are NON-ORIENTABLE.

**New Minor minimal non orientable matroids inside cubes!** - related to single-element extensions of the **cross-polytope**. (IS 2009)

**Conjecture 1.** IS 08 $Aff_{\mathbb{R}}(C^n)$ *is the unique orientable cube*

**Yes** $\Longrightarrow$ Linear/affine dependencies of $0, 1$-vectors over $\mathbb{R}$ are **purely combinatorial** : a huge net of rectangles packed in the "unique orientable way" along facets and skew-facets.

HOW DOES **Orientability** $\Longrightarrow$ **symmetry ? parallelism ? induction?**

# Orientability of Cubes I

**Theorem 1** *Every oriented cube has a canonical orientation*

An **oriented cube** = a cube **canonically oriented**

**Cocircuits complementary of the facets and skew-facets** - are the usual ones.
**Rectangles** - signed as a circuit of the real cube.

**Conjecture 1'.** *The oriented matroid $\mathcal{A}ff(\mathbb{R}^n)$ is the unique oriented cube.*

**Theorem 2** *Conjecture 1' is TRUE for $n \leq 8$.*
S. Muroga et al. 1970 - computationally verified for $n \leq 8$.
Gioan/Silva 2016 **contructive short proof for** $n \leq 7$

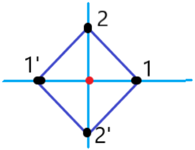
# Orientability of Cubes II -(J. Lawrence, IS 2020)

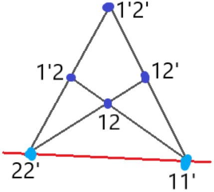
**Theorem 3** *If there are oriented cubes other then the real affine cube $Aff(C^n)$ they must be NON REPRESENTABLE (over $\mathbb{R}$).*

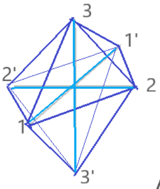
This Theorem is a consequence of the relation between CUBES and AJOINTS of the CROSS-POLYTOPE $O_n$.
**cross-polytope matroid** - $O_n$ - is the matroid of the affine dependencies of the the 2n-points of $\mathbb{R}^n$: $O \pm \mathbf{e_i}$, $\mathbf{e_i}$ - i-th vector of the canonical basis.
$O_n = \mathcal{A}ff(\{1, \ldots, n, 1', \ldots, n'\})$.

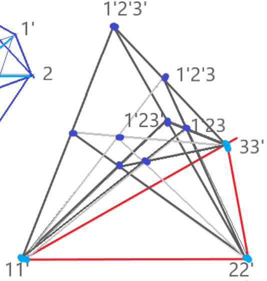
**Theorem 4** *Every oriented cube $\mathcal{M}(C^n)$ is the restriction of an adjoint of the cross-polytope matroid $O_n$.*

# Orientability of Cubes II



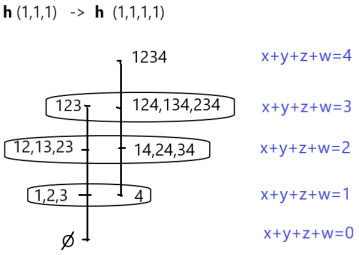




Every Cube has *n* classes of PARALLEL edges.

# The CONSTRUCTIVE proof of the unicity of the real cube for SMALL dimensions /Probability orders

- Recursive procedure of obtaining the hyperplanes of a cube $Aff(C^n)$ , from the list of hyperplanes of the cube $Aff(C^{n-1}$.
- use the rectangles to define the flats of $Aff(C^n)$ obtained by combining "stratifications" of the vertices, in two opposite facets of the cube $C^n$, orthogonal to the same vector direction $\mathbf{h} \in \mathbb{R}^{n-1}$.



# Probability orders

**Stratifications orhogonal to a vector $\iff$ probability orders** on the subsets of $[n]$:
Given $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{h} \geq \mathbf{0}$ and $\alpha, \beta \subset [n]$ $\alpha$ **is less probable then** $\beta$ iff $\mathbf{h}.\alpha < \mathbf{h}.\beta$.

Finetti 31 - axioms for qualitative probability orders q.p.o.
C.Kraft,J.Pratt, A.Seidenberg, 1959 - axioms for representable probability orders = probability orders.

**TheoremA** KPS, 1959 *A q. p. o. $\preceq$ on the substes of $[n]$ is representable if and only instead of (A), the next condition $(CC_K)$ is satisfied* **for every** *$K \in \mathbb{N}$:*

$(CC_K)$ theres is NO 2K-sequence $(\alpha_1, \ldots, \alpha_K | \beta_1, \ldots \beta_K)$ of subsets of $[n]$ with:

$\alpha_k \preceq \beta_k$ and $\alpha_k \prec \beta_k$ for some $k$
$\forall\ i \in [n]$ $|\{k : i \in \alpha_k\}| = |\{k : i \in \beta_k\}|$.

# Final Remarks:

$(CC_K)$ imposes conditions on the signed circuits of $\mathcal{A}ff(C^n)$.
Namely,
$(CC_2) \iff$ *cannonical signature of rectangles*.

Explore further!

**TheoremA'** P. Fishburn 96; D. MacLagan 99 A q. p. o. $\preceq$ on the substes of $[n]$ is representable if and only it is a representable one element extension of $Lin(\mathcal{B}_n)$ where $\mathcal{B}_n$ is the root system $\mathcal{B}_n = \{\pm e_i\} \cup \{\mathbf{e_i} \pm \mathbf{e_j}\}$.

Adjoints of the adjoint of the cross-polytope are naturally related with extensions of $Lin(\mathcal{B}_n)$. Explore further!

a recent "argument" in favour of the Conjecture:

K. Tikhomirov (2019) *$M_n$ a $n \times n$ matrix with independent $\pm 1$ entries. $P(M_n \text{ is singular}) = (1/2 + o(1))^n$*

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**THANK YOU**