Searching for solutions to Horn's problem

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The 12th Combinatorics Day

Aveiro - 21 October, 2022

The Hermitian sum eigenvalue problem
(Horn's problem)

$$\alpha = (\alpha_1, \ldots, \alpha_n)\ ,\quad \beta = (\beta_1, \ldots, \beta_n) \quad n\text{-tuples of real numbers}$$

$$\alpha_1 \geq \cdots \geq \alpha_n\ ,\quad \beta_1 \geq \cdots \geq \beta_n$$

$$\gamma = (\gamma_1, \ldots, \gamma_n)\ ,\quad \gamma_1 \geq \cdots \geq \gamma_n$$

When is γ the spectrum of $A + B$, where A and B are Hermitian with spectra α and β, respectively?

Two surveys:

W. Fulton

Eigenvalues, invariant factors, highest weights, and Schubert calculus

Bulletin AMS **37** (2000), 209-249.

R. Bhatia

Linear algebra to quantum cohomology: the story of Alfred Horn's inequalities

A.M.Monthly **108** (2001), 289-318.

Examples of valid relations

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$$

$$\gamma_6 \leq \alpha_2 + \beta_5$$

$$\gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_4 + \beta_1 + \beta_3$$

$$\gamma_3 + \gamma_5 + \gamma_9 \leq \alpha_2 + \alpha_3 + \alpha_7 + \beta_2 + \beta_4 + \beta_5$$

Notation: $E(\alpha,\beta) = \{\text{possible } \gamma\}$

Trivial: $E(\alpha,\beta)$ is contained in the hyperplane defined by the trace condition, which we abbreviate to $\Sigma\gamma = \Sigma\alpha + \Sigma\beta$.

Trivial: $E(\alpha,\beta)$ is compact, connected

(image of $\mathcal{U}_n$ under the continuous mapping $U \mapsto \lambda(D_\alpha + UD_\beta U^*)$)

Less trivial: $E(\alpha,\beta)$ is a convex polytope (Dooley+Repka+Wildberger, 1993, using symplectic geometry)

Conjecture (A. Horn, 1962):

$E(\alpha, \beta)$ is completely described by a family of inequalities of the type

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \leq \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r}$$

where $r \in \{1, ..., n\}$ and $i_1 < \ldots < i_r$, $j_1 < \ldots < j_r$, $k_1 < \ldots < k_r$.

In short,

$$\Sigma\, \gamma_K \; \leq \; \Sigma\, \alpha_I + \Sigma\, \beta_J$$

where $I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_r), K = (k_1, \ldots, k_r)$.

A consequence of this would be that $E(\alpha, \beta)$ is a convex polytope.

The question is to identify the right triples (I, J, K). Horn makes an elaborate conjecture on this, which, in sightly changed form, reads as follows.

For $I = (i_1, \ldots, i_r)$, with $1 \le i_1 < \ldots < i_r \le n$, write

$$\rho(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1)$$

Examples:

$$\rho(i) = i - 1$$

$$\rho(2, 3) = (1, 1)$$

$$\rho(3, 5, 11) = (8, 3, 2)$$

Then Horn's conjecture is:

$$\gamma \in E(\alpha, \beta)$$

$$\Updownarrow$$

$$\begin{cases} \Sigma\,\gamma = \Sigma\,\alpha + \Sigma\,\beta\,, \\ \Sigma\,\gamma_K \leq \Sigma\,\alpha_I + \Sigma\,\beta_J \ \text{ whenever} \\ \qquad\quad \rho(K) \in E[\rho(I), \rho(J)] \quad \text{(for all } r,\ 1 \leq r < n) \end{cases}$$

So E is described recursively.

This is now a theorem. (See Fulton for the long story.)

Complete solutions for $n = 1, 2, 3$

n=1

$$\gamma_1 = \alpha_1 + \beta_1$$

n=2

$$\gamma_1 \le \alpha_1 + \beta_1$$

$$\gamma_2 \le \alpha_1 + \beta_2$$

$$\gamma_2 \le \alpha_2 + \beta_1$$

$$\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$$

n=3

$$\gamma_1 \le \alpha_1 + \beta_1$$
$$\gamma_2 \le \alpha_1 + \beta_2$$
$$\gamma_3 \le \alpha_1 + \beta_3$$
$$\gamma_2 \le \alpha_2 + \beta_1$$
$$\gamma_3 \le \alpha_2 + \beta_2$$
$$\gamma_3 \le \alpha_3 + \beta_1$$
$$\gamma_1 + \gamma_2 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_2$$
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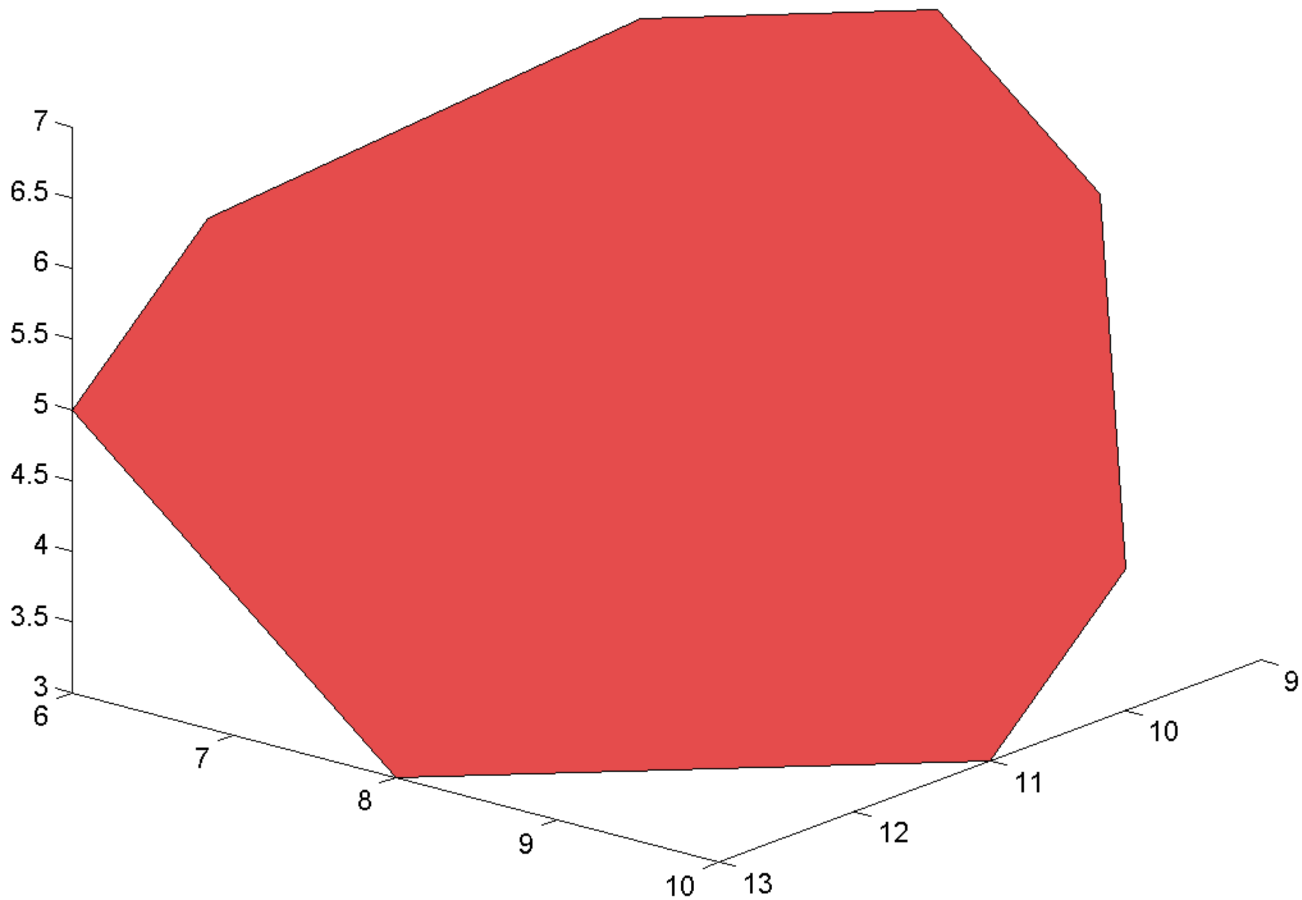
$$\gamma_2 + \gamma_3 \le \alpha_2 + \alpha_3 + \beta_1 + \beta_2$$
$$\gamma_1 + \gamma_2 + \gamma_3 = \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3$$

Example: $\alpha = (6,4,2)$, $\beta = (7,4,1)$

$$
\begin{aligned}
E(\alpha,\beta) \;=\; & \{(\gamma_1,\gamma_2,\gamma_3) : \gamma_1 \geq \gamma_2 \geq \gamma_3, \\
& \gamma_1 + \gamma_2 + \gamma_3 = 24, \\
& \gamma_1 \leq 13, \gamma_2 \leq 10, \gamma_3 \leq 7, \\
& \gamma_1 + \gamma_2 \leq 21, \gamma_1 + \gamma_3 \leq 18, \gamma_2 + \gamma_3 \leq 15\}
\end{aligned}
$$



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An open problem

Construction of solutions: Given α, β, and $\gamma \in E(\alpha, \beta)$, find Hermitian A with spectrum α and B with spectrum β such that $A + B$ has spectrum γ.

For each γ there may be many solutions.

Since the solution of Horn's problem, several authors have studied the probability distribution of γ, for given α and β.

References

Dooley+Repka+Wildberger (1993)

Frumkin+Goldberger (2006)

Zuber (2018)

Faraut (2019)

Coquereaux+McSwiggen+Zuber (2019)

Zhang+Kieburg+Forrester (2021)

Some beautiful pictures.

Only one paper – that I know of – addresses the actual construction problem:

Cao+Woerdemann (2018)

The approach is numerical.

Reduces problem to semidefinite programming and finds an algorithm that works for $n = 3$.

(The case $n = 2$ is trivial.)

An exact solution in a very particular case

W.l.g., we may assume the α's, the β's and the γ's are ≥ 0.

Take $\beta_2 = \cdots = \beta_n = 0$

So the second matrix to be constructed has rank 1.

(This covers the case where β has $n-1$ coordinates equal.)

Solved by many authors (from the 19th to the 21st century).

In this case the Horn inequalities reduce to

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1$$

$$\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \cdots \geq \gamma_n \geq \alpha_n$$

Put $D_\alpha = \text{diag}(\alpha_1, \ldots, \alpha_n)$. We are looking for a (real) column x such that $D_\alpha + xx^T$ has spectrum γ.

Denote by x^2 the column $[x_1^2 \; x_2^2 \; \cdots \; x_n^2]^T$.

Also, for each $k \in \{0, 1, \dots, n\}$, $\sigma_k(\alpha)$ is the k-th elementary symmetric function of $\alpha_1, \dots, \alpha_n$,

$$\sigma_k(\alpha) = \sum_{1 \le i_1 < \cdots < i_k \le n} \alpha_{i_1} \cdots \alpha_{i_k} \quad , \quad \sigma_0 \equiv 1$$

and we write $\sigma(\alpha)$ for the column $[\sigma_1(\alpha) \; \sigma_2(\alpha) \; \cdots \; \sigma_n(\alpha)]^T$.

Denote also by $J(\alpha)$ the Jacobian matrix of the $\sigma_k(\alpha)$, that is,

$$J(\alpha) = \left[\frac{\partial \sigma_i}{\partial \alpha_j} \right]$$

Then we can prove (JFQ, 1994) that

$$J(\alpha) \cdot x^2 = \sigma(\gamma) - \sigma(\alpha)\ .$$

We have

$$\det J(\alpha) = \prod_{i<j} (\alpha_i - \alpha_j)\ .$$

Assuming $\alpha_1 > \cdots > \alpha_n$ (w.l.g), $J(\alpha)$ is nonsingular and there is a nice expression for its inverse.

Example

$$\alpha = (6,4,2)\ ,\ \ \beta = (3,0,0)\ ,\ \ \gamma = (7,5,3)$$

We get $x = \begin{bmatrix} 0.6124 \\ 0.8660 \\ 1.3693 \end{bmatrix}$, so

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.3750 & 0.5303 & 0.8385 \\ 0.5303 & 0.7500 & 1.1859 \\ 0.8385 & 1.1859 & 1.8750 \end{bmatrix}$$

solve the problem.

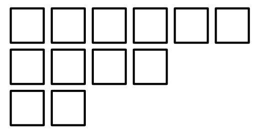
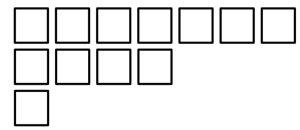
A possible general approach

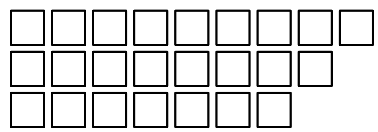
... related to the Littlewood-Richardson rule

(an object appearing in many settings, starting from representation theory)

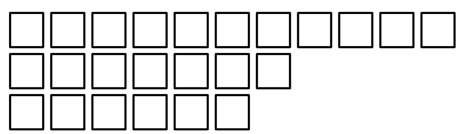
The Littlewood-Richardson rule

$\alpha = (6, 4, 2)$ $\qquad$ $\beta = (7, 4, 1)$

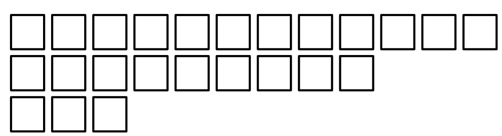
 



$(9, 8, 7)$



$(11, 7, 6)$



$(12, 9, 3)$

$(\ldots)$

$LR(\alpha, \beta)$

In the example,

$$
\begin{aligned}
LR(\alpha,\beta) \;=\; & \{(10,10,4),(11,10,3),(9,9,6),(10,9,5),(11,9,4), \\
& (12,9,3),(9,8,7),(10,8,6),(11,8,5), \\
& (12,8,4),(13,8,3),(10,7,7),(11,7,6), \\
& (12,7,5),(13,7,4),(12,6,6),(13,6,5)\}
\end{aligned}
$$

Results in 1998-1999

Santana+JFQ+Sá (1998): For integral α and β,

$$E(\alpha, \beta) \cap \mathbb{Z}^n \supseteq LR(\alpha, \beta)$$

Klyachko (1998), Knutson+Tao (1999):

$$E(\alpha, \beta) \cap \mathbb{Z}^n = LR(\alpha, \beta)$$

Example: $\alpha = (6, 4, 2)$, $\beta = (7, 4, 1)$

$$
\begin{aligned}
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& \gamma_1 + \gamma_2 + \gamma_3 = 24, \\
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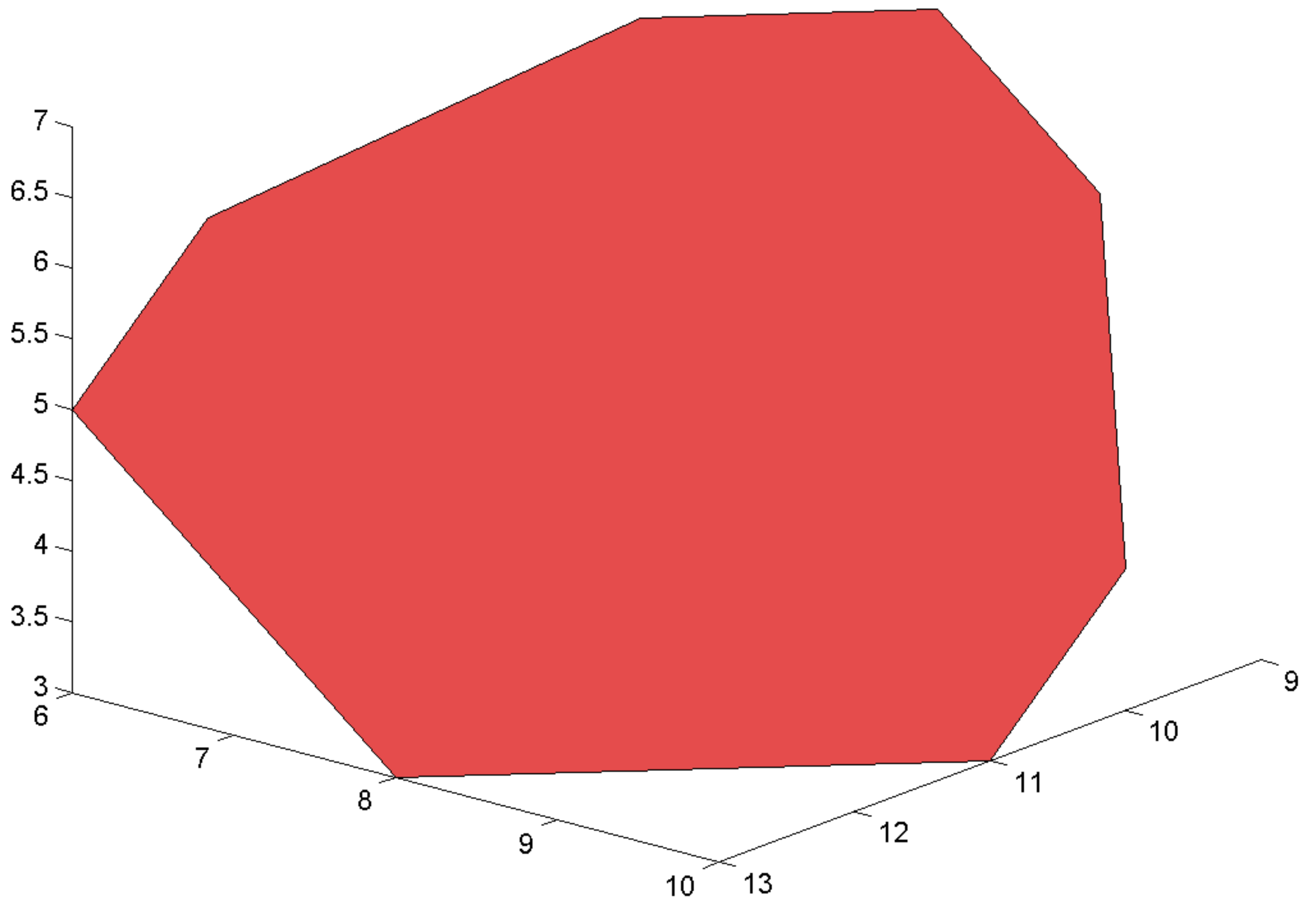
\end{aligned}
$$

$E(\alpha, \beta)$



$$
\begin{aligned}
LR(\alpha,\beta) \;=\; & \{(10,10,4),(11,10,3),(9,9,6),(10,9,5),(11,9,4),\\
& (12,9,3),(9,8,7),(10,8,6),(11,8,5),\\
& (12,8,4),(13,8,3),(10,7,7),(11,7,6),\\
& (12,7,5),(13,7,4),(12,6,6),(13,6,5)\}
\end{aligned}
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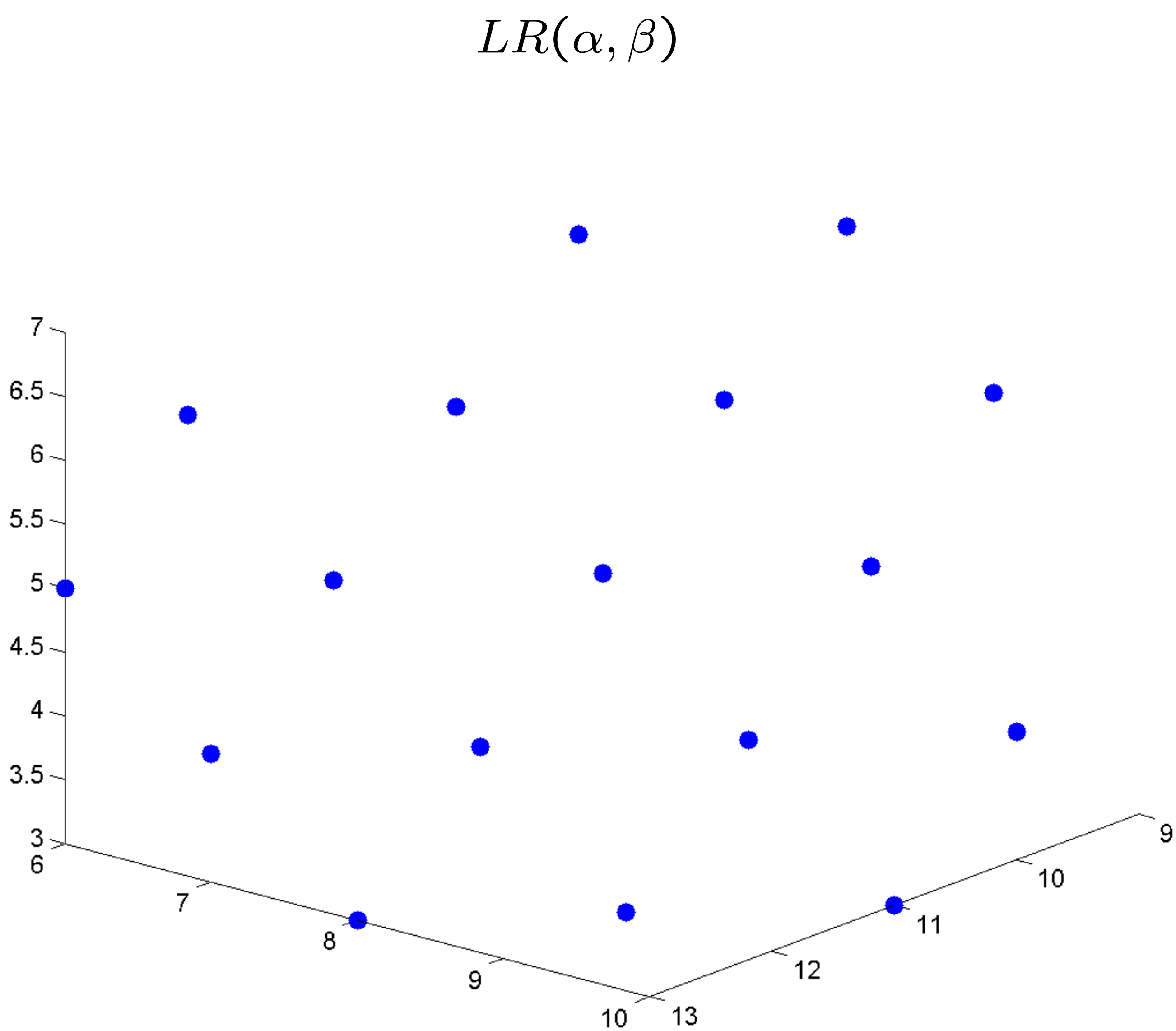

$LR(\alpha, \beta)$
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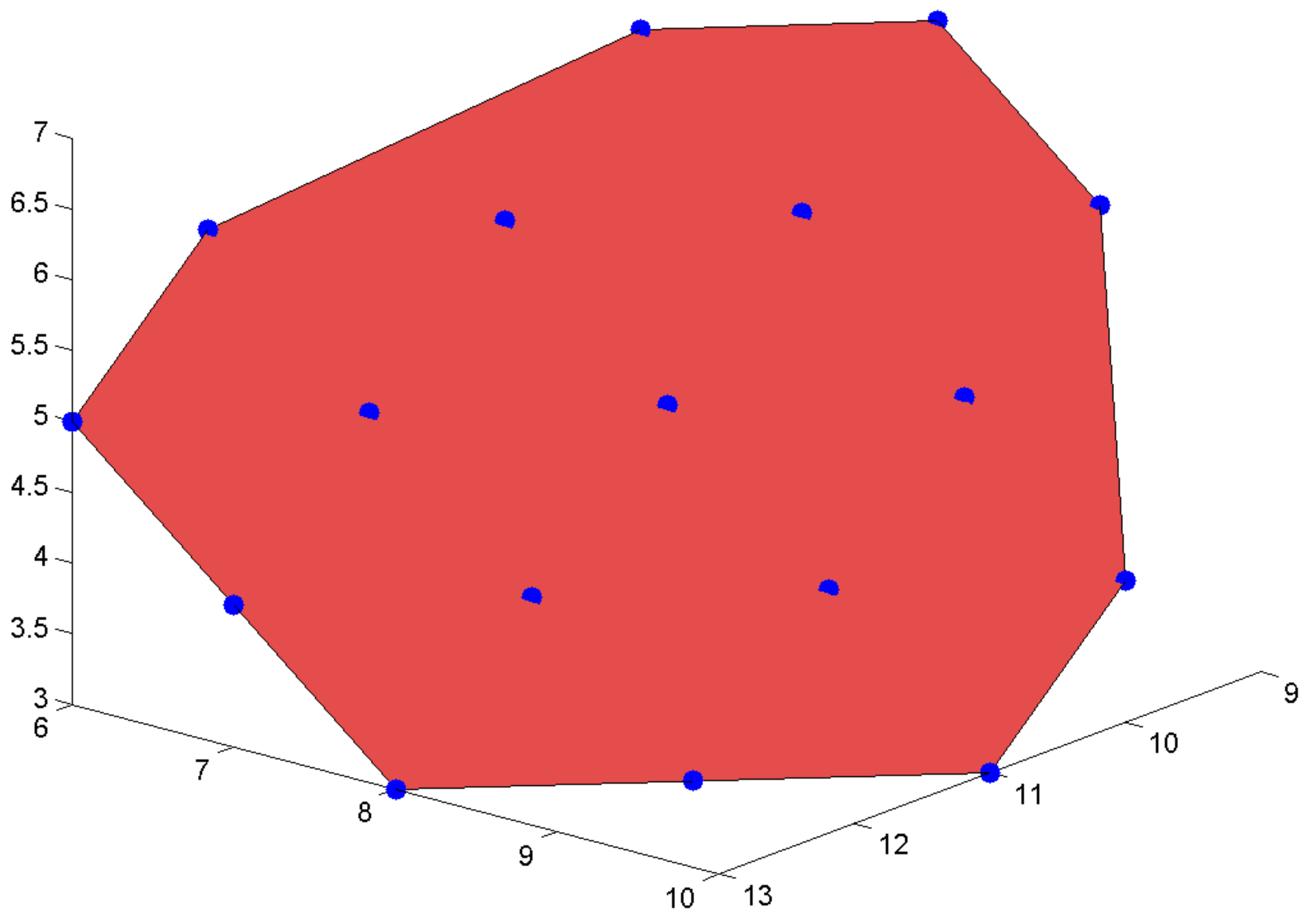
$E(\alpha, \beta)$ & $LR(\alpha, \beta)$



$0 \leq \alpha, \beta \in \mathbb{Z}^n$

$$E(\alpha, \beta) \cap \mathbb{Z}^n = LR(\alpha, \beta)$$

First, this gives an idea of why Horn's conjecture should be true, because nonempty intersections of Schubert varieties (which produce inequalities) are governed by the LR rule:

$$\Sigma\, \gamma_K \leq \Sigma\, \alpha_I + \Sigma\, \beta_J \text{ whenever}$$

$$\rho(K) \in LR[\rho(I), \rho(J)] \quad \text{(for all } r,\ 1 \leq r < n)$$

Second, it suggests a connection to another problem: invariant factors of a product of two integral matrices.

Let R be a PID (*e.g.* $\mathbb{Z}$).

$a=(a_n,\ldots,a_2,a_1)$, $b=(b_n,\ldots,b_2,b_1)$ n-tuples of nonzero elements of R

$$a_n \mid \cdots \mid a_2 \mid a_1 \ , \quad b_n \mid \cdots \mid b_2 \mid b_1$$

$$c = (c_n,\ldots,c_2,c_1) \ , \quad c_n \mid \cdots \mid c_2 \mid c_1$$

When is c the n-tuple of invariant factors of AB, where A and B have invariant factors a and b, respectively?

The Klein solution (1968)

Localization: Fix a prime $p \in R$ and work over the local ring R_p (*i.e.* work with powers of p)

$$a_i \to p^{\alpha_i},\ b_i \to p^{\beta_i},\ c_i \to p^{\gamma_i}$$

where $\alpha_1 \geq \cdots \geq \alpha_n,\ \beta_1 \geq \cdots \geq \beta_n,\ \gamma_1 \geq \cdots \geq \gamma_n$ are nonnegative integers.

Denote by $IF(\alpha, \beta)$ the set of possible γ in the invariant factor product problem.

Theorem. (Klein) $IF(\alpha, \beta) = LR(\alpha, \beta)$.

So

$$E(\alpha,\beta) \cap \mathbb{Z}^n = IF(\alpha,\beta)$$

But... there is a constructive version of Klein's theorem: Azenhas+Sá (1990)

A speculative question: is there a way of "transporting" this construction from the invariant factor setting to Hermitian matrices?

Actually, the equality $E(\alpha,\beta) \cap \mathbb{Z}^n = IF(\alpha,\beta)$ reflects a deep result, the Kirwan-Ness theorem, relating symplectic geometry to geometric invariant theory. (See Fulton's survey.)