Symplectic cacti, virtualization, and a colourful algorithm

joint work with:

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History

Cacti

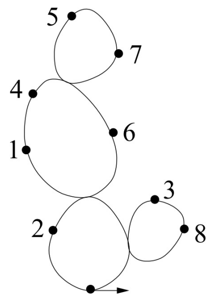


FIGURE 1. An 8 fruited cactus.

Henriques - Kamnitzer :

"Crystals and Coboundary categories"

Thm (Henriques - Kamnitzer, 2008)

$$1 \to \underbrace{\pi_1\left(\widetilde{M}^{n+1}_0\right)} \to \underbrace{J_n}_{\text{cactus group}} \to S_n \to 1$$

moduli space of n-fruited cacti.

"The pure cactus group"

cactus group

n-fruited cacti in Mexico

Notation

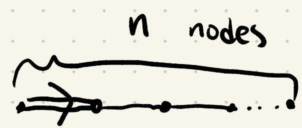
$\mathfrak{g}$ - complex semi-simple Lie algebra
$\cup$
$\mathfrak{h}$ - Cartan subalgebra D - Dynkin diagram

Φ - Root system, $\Delta = \{\alpha_i \mid i \in D\} \subseteq \Phi$ simple roots
$\cap$
$\mathfrak{h}^*_{\mathbb{R}} \supseteq \Lambda$ integral weight lattice ; $\Delta = \{\varphi \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \varphi, \alpha_i^\vee \rangle \geq 0 \ \ i \in D\}$.

$W = W(\Phi)$ Weyl group

$w_0 \in W$ longest Weyl group element

Ex : 

$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ (A_{n-1}) | 

$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ (C_n)

$\theta : D \to D$ unique automorphism such that $-w_0 \alpha_i = \alpha_{\theta(i)}$ for all $i \in [1, n]$.

Ex : A_{n-1} : $\theta(d) = n - d$ C_n : $\theta(d) = d$.

Crystal Basics

A (seminormal) $\mathfrak{g}$- crystal is a finite set B along with maps

$\mathrm{wt}: B \longrightarrow \Lambda$

$e_i, f_i: B \to B \sqcup \{0\}$ ← the "root" operators ... inverse to each other.

$\varepsilon_i, \varphi_i: B \to \mathbb{Z}_{\geq 0} \quad \forall i \in D.$

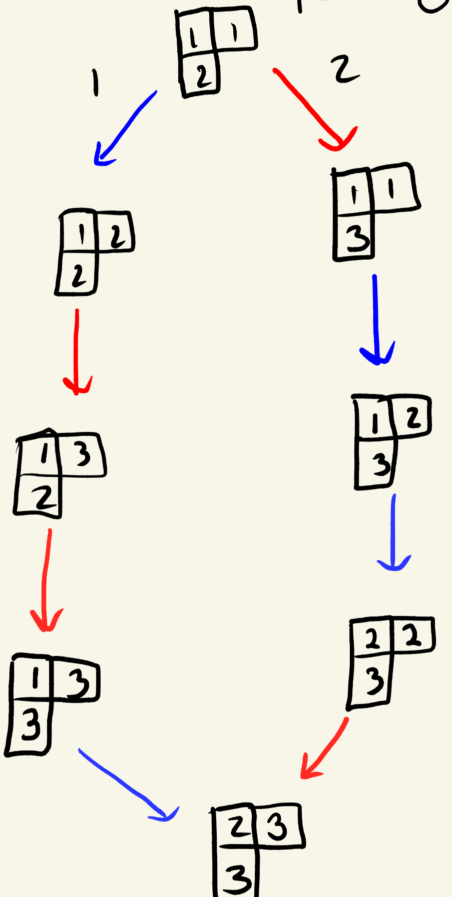
satisfying certain axioms.

A highest weight $\mathfrak{g}$- crystal is a seminormal $\mathfrak{g}$-crystal $B(\lambda)$ that is connected, and has a unique $u_\lambda \in B(\lambda)$ which generates $B(\lambda)$ and has weight $\mathrm{wt}(u_\lambda) = \lambda$.

Idea

$$\left\{ \begin{array}{c} \text{highest weight} \\ \mathfrak{g}\text{- crystals} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{finite - dimensional,} \\ \text{irreducible} \\ \mathfrak{g}\text{- representations} \end{array} \right\}$$

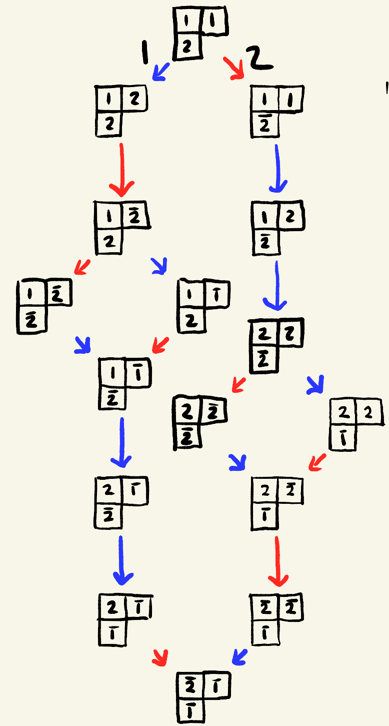
Highest weight tableau $\mathfrak{g}$-crystals for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$



A_2

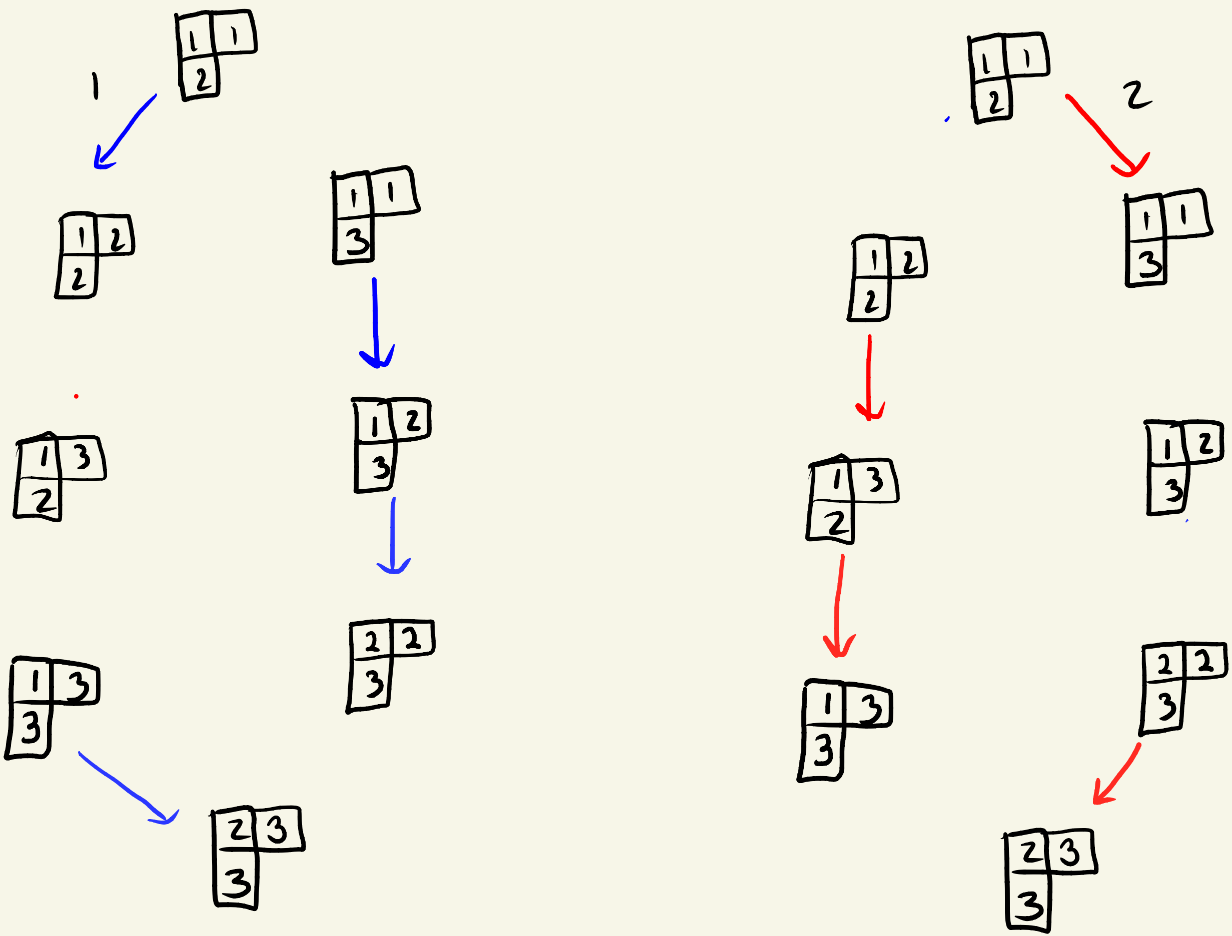
"semistandard young tableaux"

C_2



"Kashiwara-Nakashima (KN) tableaux"

Levi branching of crystals B_J for $J \subseteq D$



Motivation

A, B $\mathfrak{g}$-crystals. The map defined by $\begin{array}{rcl} A\otimes B & \to & B\otimes A \\ a\otimes b & \mapsto & b\otimes a \end{array}$ is not a crystal morphism.

Idea (of Berenstein): to define an involution $\xi_B : B \to B$ for any $\mathfrak{g}$-crystal B such that for $\mathfrak{g}$-crystals A, B,

$$(a,b) \longmapsto \xi_{A\otimes B}\left(\xi_B(b), \xi_A(a)\right)$$

is an isomorphism of crystals.

Schützenberger-Lusztig (KL) involution

$B(\lambda)$ - normal $\mathfrak{g}$-crystal of h.w. λ.

u_λ^{high}, u_λ^{low} highest, respectively lowest weight elements of $B(\lambda)$.

The Schützenberger-Lusztig involution $\xi : B(\lambda) \to B(\lambda)$ is the unique set involution s.th

$$e_i \xi(b) = \xi f_{\theta(i)}(b)$$

$$f_i \xi(b) = \xi e_{\theta(i)}(b)$$

$$wt(\xi(b)) = w_0 wt(b)$$

In particular,

$$\xi(u_\lambda^{high}) = u_\lambda^{low} \quad \text{and}$$

$$\xi(u_\lambda^{low}) = u_\lambda^{high}.$$

If $b = f_{j_r} \cdots f_{j_1}(u_\lambda^{high})$, $\xi(b) = e_{\theta(j_r)} \cdots e_{\theta(j_1)}(u^{low})$

KL-involution on tableau crystals

- SSYT well known as Schützenberger involution on semi-standard Young tableaux.

- Kashiwara-Nakashima tableaux - work by Santos!

The cactus group

Definition (Halacheva)

The cactus group J_D is defined by:

generators: s_I, $I \subseteq D$ connected sub-diagram

relations: $s_I^2 = 1$

$s_I s_J = s_J s_I$ if $I \sqcup J$ is disconnected

$s_I s_J = s_{\theta_I(J)} s_I$ if $J \subseteq I$.

Remark: This generalizes the definition of Henriques-Kamnitzer, who originally worked in type A only.

Action on $\mathfrak{g}$-crystals

Theorem (Halacheva)

Let B be a normal $\mathfrak{g}$-crystal. Then $J_{\mathfrak{g}}$ acts on B, by

$$s_I \longmapsto \xi_I \text{ on } B_I \leftarrow \text{Levi branching of } B.$$

↑ the "Levi-branched" Schützenberger-Lusztig involution.

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $B = SSYT(\lambda)$, $I = [1, i] \subseteq [1, n]$:

- freeze entries in I in T.
- jdt them out
- do Schützenberger involution
- jdt the outer entries back in.

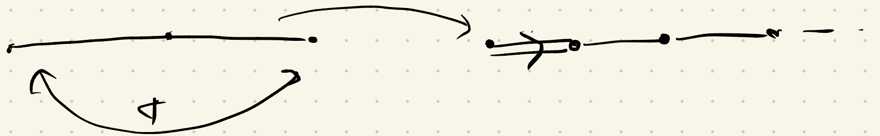
The Colourful Algorithm

Algorithm for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $\mathcal{B} = KL(\lambda, n)$, $I = [i, n] \subseteq [1, n]$

[Azenhas - Tanghat - T] $\quad T \in KL(\lambda, n)$

- Freeze letters in $I \rightsquigarrow T_I$. Green
- If T_I is not KL, apply "contraction relation" (R3) to each column purple
- Apply sjdt to colourful letters $\longrightarrow$ some new Red letters may appear
- Do SL to what you get
- $SJDT^{-1}$ to the colourful letters $\longrightarrow$ Done!

Virtualization



A_{2n-1}

SSYT of shape λ^A in the ordered alphabet $\{1 < \cdots < n < \bar{n} < \cdots \bar{1}\}$

On tableaux:

Baker's virtualization is an injective map

$$KN(\lambda, n) \xrightarrow{E} SSYT(\lambda^A, n, \bar{n})$$

such that $B(KN(\lambda,n))$, $\tilde{f}_i = f_i \circ f_{\overline{i+1}}$, $\tilde{e}_i = e_i \circ e_{\overline{i+1}}$ $\quad \tilde{e}_n = e_n^2$, $\tilde{f}_n = f_n^2$

has the structure of a crystal isomorphic to $KN(\lambda, n)$.

Theorem (Azenhas–Tarighat-Feller–T, 2022) Let $J \subseteq [1,n]$.

1. The following diagram commutes:

$$\begin{tikzcd}
KN(\lambda, n) \arrow[r, "E"] \arrow[d, "\xi_J^{C_n}"'] & SSYT(\lambda^A, n, \bar{n}) \arrow[d, "\xi^{A_{2n-1}}_{[p,\overline{p+1}]} \,/\, \xi^{A_{2n-1}}_{[p,q]\cup[\overline{q+1},\overline{p+1}]} = \xi^{A_{2n-1}}_{[p,q]}\xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}"] \\
KN(\lambda, n) \arrow[r, "E"] & SSYT(\lambda^A, n, \bar{n})
\end{tikzcd}$$

$J = [p,n]$ or $J = [p,q]$

Moreover, the left inverse E^{-1} can be computed explicitly.

2. There is a group monomorphism

$$J_{sp(2n,\mathbb{C})} \longrightarrow J_{sl(2n,\mathbb{C})}$$

given by:

$$s_{[p,n]} \longmapsto s_{[p,\overline{p+1}]}$$

$$1 \le p < q < n \qquad s_{[p,q]} \longmapsto s_{[p,q]}\, s_{[\overline{q+1},\overline{p+1}]}$$