Generalizing the hypoplactic monoid through quasi-crystal graphs

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Outline

Crystals and the plactic monoid

- Lascoux and Schützenberger (1981) introduced the classical plactic monoid based on work by Schensted (1961) and Knuth (1970).
- Kashiwara (1990) introduced crystal bases for the vector representation of the quantized universal general linear Lie algebra, from where the classical plactic monoid arises by identifying elements in isomorphic connected components of the resulting crystal graph.
- Kashiwara and Nakashima (1994) studied the crystal graphs for Cartan types $\mathfrak{B}_n$, $\mathfrak{C}_n$ and $\mathfrak{D}_n$.
- Lecouvey (2002, 2003, 2007) made an in-depth study of the plactic monoids for Cartan types $\mathfrak{B}_n$, $\mathfrak{C}_n$ and $\mathfrak{D}_n$.

Example 1.

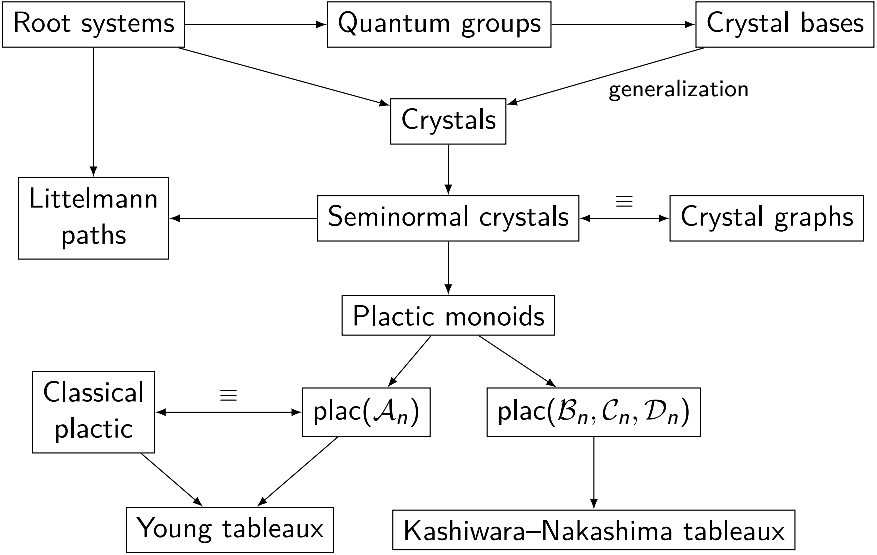
- By computing the Young tableaux we have that

$$121 \approx 112, \qquad \text{as} \qquad \mathrm{P}(121) = \mathrm{P}(112) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

- The following components of the crystal graph of $\mathcal{A}_3^{\tilde{*}}$ are isomorphic.

$$\begin{tikzcd}
 & & 112 \arrow[dl, "1"'] \arrow[dr, "2"] & & \\
 & 212 \arrow[dl, "2"'] & & 113 \arrow[dr, "1"] & \\
312 \arrow[dr, "2"'] & & & & 213 \arrow[dl, "1"] \\
 & 313 \arrow[dr, "1"'] & & 223 \arrow[dl, "2"] & \\
 & & 323 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 121 \arrow[dl, "1"'] \arrow[dr, "2"] & & \\
 & 122 \arrow[dl, "2"'] & & 131 \arrow[dr, "1"] & \\
132 \arrow[dr, "2"'] & & & & 231 \arrow[dl, "1"] \\
 & 133 \arrow[dr, "1"'] & & 232 \arrow[dl, "2"] & \\
 & & 233 & &
\end{tikzcd}$$

Crystals and the plactic monoid



Quasi-crystals and the hypoplactic monoid

- Krob and Thibon (1997) obtained the classical hypoplactic monoid through representation-theoretical interpretations of quasi-symmetric functions and noncommutative symmetric functions.
- Novelli (2000) made a combinatorial study of the classical hypoplactic monoid. Also, analogues of results for the classical plactic monoid are proven for the hypoplactic monoid.
- Cain and Malheiro (2017) obtained the classical hypoplactic monoid by identifying elements in isomorphic components of a quasi-crystal graph derived from the crystal graph of $\mathcal{A}_n^{\tilde{*}}$.

Example 2.

- From the quartic relations we have that

$$1212 \mathrel{\ddot{\sim}} 2121, \qquad 2123 \mathrel{\ddot{\sim}} 1223, \qquad 2313 \mathrel{\ddot{\sim}} 3231.$$

- By computing the quasi-ribbon tableaux, we get that

$$\mathrm{QR}(1212) = \mathrm{QR}(2121) = \begin{ytableau} 1 & 1 \\ \none & 2 & 2 \end{ytableau},$$

$$\mathrm{QR}(2132) = \mathrm{QR}(1223) = \begin{ytableau} 1 \\ 2 & 2 \\ \none & 3 \end{ytableau},$$

$$\mathrm{QR}(2313) = \mathrm{QR}(3231) = \begin{ytableau} 1 & 2 \\ \none & 3 & 3 \end{ytableau}.$$

Quasi-crystals and the hypoplactic monoid

The quasi-crystal graph of rank 3 is obtained from the crystal graph of $\mathcal{A}_3^{\tilde{*}}$ by removing edges labelled by i starting or ending on a word of the form $w_1 i w_2 (i+1) w_3$.

Example 2 (cont.).

- The following components of the quasi-crystal graph are isomorphic.

$$\begin{tikzcd}
2121 \arrow[dr, "2"] & \\
& 3121 \arrow[dl, "2"'] \\
3131 \arrow[dr, "1"] & \\
& 3231 \arrow[dl, "1"'] \\
3232 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1212 \arrow[dr, "2"] & \\
& 1312 \arrow[dl, "2"'] \\
1313 \arrow[dr, "1"] & \\
& 2313 \arrow[dl, "1"'] \\
2323 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1122 \arrow[dr, "2"] & \\
& 1132 \arrow[dl, "2"'] \\
1133 \arrow[dr, "1"] & \\
& 2133 \arrow[dl, "1"'] \\
2233 &
\end{tikzcd}$$

Remark 1 (Remark 6.17, p. 70).

1. Consider the following components of the crystal graph of $\mathcal{C}_2^{\tilde{*}}$.

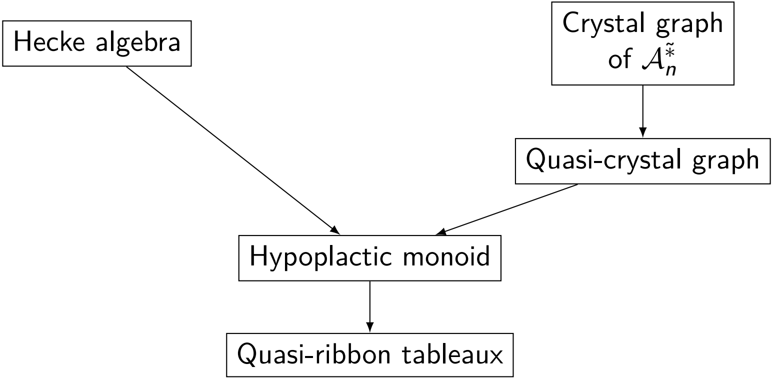
$$\begin{array}{ccccccccc} \epsilon & \quad & 1 & \xrightarrow{1} & 2 & \xrightarrow{2} & \overline{2} & \xrightarrow{1} & \overline{1} \\ 1\overline{1} & \quad & 1\overline{1}1 & \xrightarrow{1} & 1\overline{1}2 & \xrightarrow{2} & 1\overline{1}\overline{2} & \xrightarrow{1} & 1\overline{1}\overline{1} \end{array}$$

2. By removing the edges labelled by i starting or ending on a word of the form $w_1 i w_2 (i+1) w_3$, we get the following components.

$$\begin{array}{ccccccccc} \epsilon & \quad & 1 & \xrightarrow{1} & 2 & \xrightarrow{2} & \overline{2} & \xrightarrow{1} & \overline{1} \\ 1\overline{1} & \quad & 1\overline{1}1 & & 1\overline{1}2 & \xrightarrow{2} & 1\overline{1}\overline{2} & \xrightarrow{1} & 1\overline{1}\overline{1} \end{array}$$

3. Identifying elements in isomorphic components does **not** result in a monoid congruence, because $\epsilon \sim 1\overline{1}$ and $1 \sim 1$, but $1\overline{1}1 \not\sim 1$.
4. The method by Cain and Malheiro (2017) does not work for type $\mathfrak{C}_2$.

Quasi-crystals and the hypoplactic monoid



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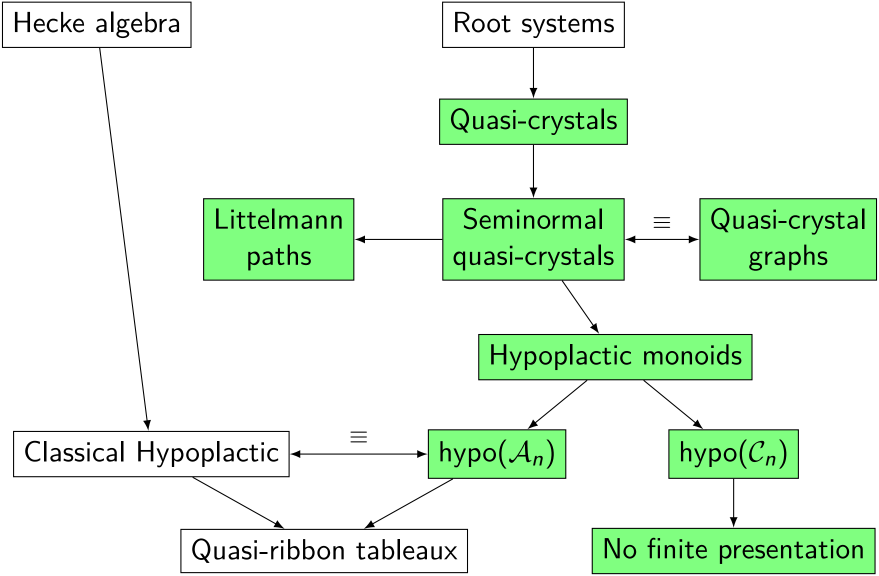


Underlying algebraic structure

- V a Euclidean space with inner product $\langle \cdot, \cdot \rangle$.
- Φ a root system.
- Λ a weight lattice.
- I index set for the simple roots $(\alpha_i)_{i \in I}$.

Example 3.

The root system for Cartan type $\mathfrak{A}_n$:

$$\Phi = \{\mathbf{e_i} - \mathbf{e_j} \mid i \neq j\},$$

$\Lambda = \mathbb{Z}^n$, and $\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}$, $i = 1, 2, \ldots, n-1$.

The root system for Cartan type $\mathfrak{C}_n$:

$$\Phi = \{\pm\mathbf{e_i} \pm \mathbf{e_j} \mid i < j\} \cup \{\pm 2\mathbf{e_i} \mid i = 1, 2, \ldots, n\},$$

$\Lambda = \mathbb{Z}^n$, and $\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}$, $i = 1, 2, \ldots, n-1$, and $\alpha_n = 2\mathbf{e_n}$.

A general notion of quasi-crystals

Definition 2 (Definition 7.1, p. 72).

A **quasi-crystal** $\mathcal{Q}$ of type Φ consists of a set Q, and maps $\mathrm{wt} : Q \to \Lambda$, $\ddot{e}_i, \ddot{f}_i : Q \to Q \sqcup \{\bot\}$ and $\ddot{\varepsilon}_i, \ddot{\varphi}_i : Q \to \mathbb{Z} \cup \{-\infty, +\infty\}$, $i \in I$, satisfying:

1. $\ddot{\varphi}_i(x) = \ddot{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$;
2. if $\ddot{e}_i(x) \in Q$, then $\mathrm{wt}(\ddot{e}_i(x)) = \mathrm{wt}(x) + \alpha_i$, $\ddot{\varepsilon}_i(\ddot{e}_i(x)) = \ddot{\varepsilon}_i(x) - 1$, and $\ddot{\varphi}_i(\ddot{e}_i(x)) = \ddot{\varphi}_i(x) + 1$;
3. if $\ddot{f}_i(x) \in Q$, then $\mathrm{wt}(\ddot{f}_i(x)) = \mathrm{wt}(x) - \alpha_i$, $\ddot{\varepsilon}_i(\ddot{f}_i(x)) = \ddot{\varepsilon}_i(x) + 1$, and $\ddot{\varphi}_i(\ddot{f}_i(x)) = \ddot{\varphi}_i(x) - 1$;
4. $\ddot{e}_i(x) = y$ if and only if $x = \ddot{f}_i(y)$;
5. if $\ddot{\varepsilon}_i(x) = -\infty$ then $\ddot{e}_i(x) = \ddot{f}_i(x) = \bot$;
6. if $\ddot{\varepsilon}_i(x) = +\infty$ then $\ddot{e}_i(x) = \ddot{f}_i(x) = \bot$.

Seminormal quasi-crystals and homomorphisms

A quasi-crystal $\mathcal{Q}$ is **seminormal** if

1. $\ddot{\varepsilon}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \ddot{e}_i^k(x) \in Q\}$, and
2. $\ddot{\varphi}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \ddot{f}_i^k(x) \in Q\}$,

whenever $\ddot{\varepsilon}_i(x) \neq +\infty$.

Definition 3 (Definition 7.12, p. 77).

A **quasi-crystal homomorphism** $\psi : \mathcal{Q} \to \mathcal{Q}'$ is a map $\psi : Q \sqcup \{\perp\} \to Q' \sqcup \{\perp\}$ satisfying:

1. $\psi(\perp) = \perp$;
2. if $\psi(x) \in Q'$, then $\mathrm{wt}(\psi(x)) = \mathrm{wt}(x)$, $\ddot{\varepsilon}_i(\psi(x)) = \ddot{\varepsilon}_i(x)$, and $\ddot{\varphi}_i(\psi(x)) = \ddot{\varphi}_i(x)$;
3. if $\ddot{e}_i(x) \in Q$ and $\psi(x), \psi(\ddot{e}_i(x)) \in Q'$, then $\psi(\ddot{e}_i(x)) = \ddot{e}_i(\psi(x))$;
4. if $\ddot{f}_i(x) \in Q$ and $\psi(x), \psi(\ddot{f}_i(x)) \in Q'$, then $\psi(\ddot{f}_i(x)) = \ddot{f}_i(\psi(x))$.

Quasi-crystal graphs

Definition 4 (Definition 7.19, p. 81).

The **quasi-crystal graph** $\Gamma_{\mathcal{Q}}$ of a quasi-crystal $\mathcal{Q}$ is a Λ-weighted I-labelled directed graph with:

- vertex set Q;
- an edge $x \xrightarrow{i} y$, if $\ddot{f}_i(x) = y$;
- a loop $x \circlearrowright i$, if $\ddot{\varepsilon}_i(x) = +\infty$.

Theorem 5 (Proposition 7.27 and Remark 7.28, pp. 85–86).

A seminormal quasi-crystal is completely determined by its quasi-crystal graph.

Remark 6 (Remark 7.20, p. 81).

The quasi-crystal graph of a crystal is a crystal graph.

The class of seminormal quasi-crystal graphs

Consider a root system Φ with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$.

A Λ-weighted I-labelled directed graph Γ is a **seminormal quasi-crystal graph** if for any vertices x and y, and any $i \in I$, the following conditions are satisfied:

1. x is the start of at most an edge labelled by i, and is the end of at most an edge labelled by i;
2. any i-labelled path of Γ is finite;
3. if $x \xrightarrow{i} y$ is an edge of Γ with $x \neq y$, then $\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha_i$;
4. $\ddot{\varphi}_i(x) = \ddot{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$, where
    - $\ddot{\varphi}_i(x)$ is the supremum among nonnegative integers $k \in \mathbb{Z}_{\geq 0}$ such that there exists an i-labelled walk on Γ starting on x with length k,
    - $\ddot{\varepsilon}_i(x)$ is the supremum among nonnegative integers $l \in \mathbb{Z}_{\geq 0}$ such that there exists an i-labelled walk on Γ ending on x with length l.

The class of **seminormal crystal graphs** corresponds to the class of seminormal quasi-crystal graphs that are simple.

Definition 7 (Theorem 7.32 and Definition 7.33, pp. 88–92).

Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystals of a same type. The **inverse-free quasi-tensor product** $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is a seminormal quasi-crystal consisting of the set of ordered pairs $Q \ddot{\otimes} Q'$ and quasi-crystal structure given by:

1. $\mathrm{wt}(x \ddot{\otimes} x') = \mathrm{wt}(x) + \mathrm{wt}(x')$;
2. if $\ddot{\varphi}_i(x) > 0$ and $\ddot{\varepsilon}_i(x') > 0$, set $\ddot{e}_i(x \ddot{\otimes} x') = \ddot{f}_i(x \ddot{\otimes} x') = \perp$ and $\ddot{\varepsilon}_i(x \ddot{\otimes} x') = \ddot{\varphi}_i(x \ddot{\otimes} x') = +\infty$;
3. otherwise, set

$$\ddot{e}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{e}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) \geq \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{e}_i(x') & \text{if } \ddot{\varphi}_i(x) < \ddot{\varepsilon}_i(x'), \end{cases}$$

$$\ddot{f}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{f}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) > \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{f}_i(x') & \text{if } \ddot{\varphi}_i(x) \leq \ddot{\varepsilon}_i(x'), \end{cases}$$

$$\ddot{\varepsilon}_i(x \ddot{\otimes} x') = \max\{\ddot{\varepsilon}_i(x), \ddot{\varepsilon}_i(x') - \langle \mathrm{wt}(x), \alpha_i^\vee \rangle\},$$

$$\ddot{\varphi}_i(x \ddot{\otimes} x') = \max\{\ddot{\varphi}_i(x) + \langle \mathrm{wt}(x'), \alpha_i^\vee \rangle, \ddot{\varphi}_i(x')\},$$

Quasi-crystal graphs of quasi-tensor products

The quasi-crystal graph $\Gamma_{\mathcal{Q} \ddot{\otimes} \mathcal{Q}'}$ of a quasi-tensor product $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is obtained as follows:

1. the vertex set is $Q \ddot{\otimes} Q'$ and the weight map is defined by $\mathrm{wt}(x \ddot{\otimes} y) = \mathrm{wt}(x) + \mathrm{wt}(y)$;
2. an i-labelled loop $x \ddot{\otimes} y \circlearrowleft i$, whenever
   - x has an i-labelled loop, or
   - y has an i-labelled loop, or
   - x is the start of an i-labelled edge and y is the end of an i-labelled edge;
3. an i-labelled edge $x \ddot{\otimes} y \xrightarrow{i} u \ddot{\otimes} y$, whenever
   - $x \xrightarrow{i} u$ and y is not the end of an i-labelled edge, that is, $v \not\xrightarrow{i} y$ for any $v \in Q'$;
4. an i-labelled edge $x \ddot{\otimes} y \xrightarrow{i} x \ddot{\otimes} v$, whenever
   - $y \xrightarrow{i} v$ and x is not the start of an i-labelled edge, that is, $x \not\xrightarrow{i} u$ for any $u \in Q$.

Example 4.

The quasi-crystal graphs of $\mathcal{C}_2$ and $\mathcal{C}_2 \mathbin{\ddot{\otimes}} \mathcal{C}_2$ are

$$
\begin{tikzcd}
1 \arrow[r, "1"] & 2 \arrow[r, "2"] & \overline{2} \arrow[r, "1"] & \overline{1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \ddot{\otimes} 1 \arrow[r, "1"] & 2 \ddot{\otimes} 1 \arrow[r, "2"] \arrow[d, "1"] & \overline{2} \ddot{\otimes} 1 \arrow[r, "1"] & \overline{1} \ddot{\otimes} 1 \arrow[d, "1"] \\
1 \ddot{\otimes} 2 \arrow[loop left, "1"] \arrow[d, "2"] & 2 \ddot{\otimes} 2 \arrow[r, "2"] & \overline{2} \ddot{\otimes} 2 \arrow[loop right, "1"] \arrow[d, "2"] & \overline{1} \ddot{\otimes} 2 \arrow[d, "2"] \\
1 \ddot{\otimes} \overline{2} \arrow[r, "1"] & 2 \ddot{\otimes} \overline{2} \arrow[loop right, "2"] \arrow[d, "1"] & \overline{2} \ddot{\otimes} \overline{2} \arrow[r, "1"] & \overline{1} \ddot{\otimes} \overline{2} \arrow[d, "1"] \\
1 \ddot{\otimes} \overline{1} \arrow[loop left, "1"] & 2 \ddot{\otimes} \overline{1} \arrow[r, "2"] & \overline{2} \ddot{\otimes} \overline{1} \arrow[loop right, "1"] & \overline{1} \ddot{\otimes} \overline{1}
\end{tikzcd}
$$

with weights $\mathrm{wt}(1) = \mathbf{e_1}$, $\mathrm{wt}(2) = \mathbf{e_2}$, $\mathrm{wt}(\overline{2}) = -\mathbf{e_2}$, $\mathrm{wt}(\overline{1}) = -\mathbf{e_1}$, and $\mathrm{wt}(x \mathbin{\ddot{\otimes}} y) = \mathrm{wt}(x) + \mathrm{wt}(y)$.

Signature rule for quasi-tensor product

Theorem 8 (Theorem 7.37, p. 94).

The quasi-tensor product is associative, i.e.
$(\mathcal{Q}_1 \ddot{\otimes} \mathcal{Q}_2) \ddot{\otimes} \mathcal{Q}_3 \cong \mathcal{Q}_1 \ddot{\otimes} (\mathcal{Q}_2 \ddot{\otimes} \mathcal{Q}_3)$.

Consider the zero monoid $Z_0 = \langle -, + \mid +- = 0 \rangle$.

Definition 9.

Let $\mathcal{Q}$ be a seminormal quasi-crystal. For each $i \in I$, the ***i*-signature map for the quasi-tensor product** $\operatorname{sgn}_i^{\ddot{\otimes}} : Q \to Z_0$ is given by

$$\operatorname{sgn}_i^{\ddot{\otimes}}(x) = \begin{cases} 0 & \text{if } \ddot{\varepsilon}_i(x) = +\infty \\ -^{\ddot{\varepsilon}_i(x)} +^{\ddot{\varphi}_i(x)} & \text{otherwise,} \end{cases}$$

Theorem 10 (Proposition 7.40, p. 99).

Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystals of the same type. Then,

$$\operatorname{sgn}_i^{\ddot{\otimes}}(x \ddot{\otimes} x') = \operatorname{sgn}_i^{\ddot{\otimes}}(x) \operatorname{sgn}_i^{\ddot{\otimes}}(x'),$$

Quasi-crystal monoids

Definition 11 (Definition 8.1, p. 103).

A **quasi-crystal monoid** $\mathcal{M}$ of type Φ consists of

1. a set M;
2. a seminormal quasi-crystal with underlying set M;
3. a monoid with underlying set M;
4. the map $x \mathbin{\ddot{\otimes}} y \mapsto xy$ is a quasi-crystal homomorphism from $\mathcal{M} \mathbin{\ddot{\otimes}} \mathcal{M}$ to $\mathcal{M}$.

Definition 12 (Definition 8.7, p. 109).

Let $\mathcal{M}$ and $\mathcal{M}'$ be quasi-crystal monoids of the same type. A **quasi-crystal monoid homomorphism** $\psi : \mathcal{M} \to \mathcal{M}'$, is a map $\psi : M \to M'$ which is both a quasi-crystal and a monoid homomorphism.

Quasi-crystal graphs of quasi-crystal monoids

The quasi-crystal graph $\Gamma_{\mathcal{M}}$ of a quasi-crystal monoid $\mathcal{M}$ satisfies:

1. the vertex set is a monoid M and the weight map is defined by $\mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y)$;
2. an i-labelled loop $xy \circlearrowleft_i$, whenever
   - x has an i-labelled loop, or
   - y has an i-labelled loop, or
   - x is the start of an i-labelled edge and y is the end of an i-labelled edge;
3. an i-labelled edge $xy \xrightarrow{i} uy$, whenever
   - $x \xrightarrow{i} u$ and y is not the end of an i-labelled edge, that is, $v \not\xrightarrow{i} y$ for any $v \in Q'$;
4. an i-labelled edge $xy \xrightarrow{i} xv$, whenever
   - $y \xrightarrow{i} v$ and x is not the start of an i-labelled edge, that is, $x \not\xrightarrow{i} u$ for any $u \in Q$.

Signature rule for quasi-crystal monoids

Theorem 13 (Proposition 8.4, p. 107).

Let $\mathcal{M}$ be a quasi-crystal monoid. Then, for each $i \in I$, either

- $\ddot{\varepsilon}_i(1) = \ddot{\varphi}_i(1) = 0$; *or*
- $\ddot{\varepsilon}_i(x) = \ddot{\varphi}_i(x) = +\infty$, *for all* $x \in M$.

Definition 14 (p. 107).

A quasi-crystal monoid is **nondegenerate** if $\ddot{\varepsilon}_i(1) = 0$, for all $i \in I$.

Theorem 15 (Proposition 8.5, p. 108).

Let $\mathcal{M}$ be a nondegenerate quasi-crystal monoid. Then, for each $i \in I$, the i-signature map $\operatorname{sgn}_i^{\ddot{\otimes}}$ *is a monoid homomorphism.*

Free quasi-crystal monoid

A detailed description of the free quasi-crystal monoid over a seminormal quasi-crystal is done in Definition 8.8 (p. 111). As the term **free** suggests it can also be defined (up to isomorphism) by the following universal property.

Theorem 16 (Theorem 8.11 and Corollary 8.12, pp. 113–114).

Let $\mathcal{Q}$ be a seminormal quasi-crystal. The free quasi-crystal monoid $\mathcal{Q}^{\ddot{*}}$ over $\mathcal{Q}$ is the unique quasi-crystal monoid such that for any nondegenerate quasi-crystal monoid $\mathcal{M}$ and any quasi-crystal homomorphism $\psi : \mathcal{Q} \to \mathcal{M}$ with $\psi(Q) \subseteq M$, there exists a unique quasi-crystal monoid homomorphism $\hat{\psi} : \mathcal{Q}^{\ddot{*}} \to \mathcal{M}$ for which the following diagram commutes

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "\psi"] \arrow[dr, "\iota"'] & & \mathcal{M} \\ & \mathcal{Q}^{\ddot{*}} \arrow[ur, dashed, "\hat{\psi}"'] & \end{tikzcd}$$

where ι denotes the inclusion map.

Quasi-crystal graphs of free quasi-crystal monoids

The quasi-crystal graph $\Gamma_{\mathcal{Q}^{\ddot{*}}}$ of a free quasi-crystal monoid $\mathcal{Q}^{\ddot{*}}$ over $\mathcal{Q}$ consists of the vertex set Q^* and is inductively constructed for $x \in Q$ and $w \in Q^*$ as follows:

1. $\mathrm{wt}(wx) = \mathrm{wt}(w) + \mathrm{wt}(x)$;
2. an i-labelled loop $wx \circlearrowright i$, whenever
   - w has an i-labelled loop, or
   - x has an i-labelled loop, or
   - w is the start of an i-labelled edge and x is the end of an i-labelled edge;
3. an i-labelled edge $wx \xrightarrow{i} ux$, whenever
   - $w \xrightarrow{i} u$ and x is not the end of an i-labelled edge;
4. an i-labelled edge $wx \xrightarrow{i} wy$, whenever
   - $x \xrightarrow{i} y$ and w is not the start of an i-labelled edge.

Congruences on quasi-crystal monoids

Definition 17 (Definition 8.13, p. 115).

Let $\mathcal{M}$ be a quasi-crystal monoid. A **quasi-crystal monoid congruence** on $\mathcal{M}$ is a equivalence relation $\theta \subseteq M \times M$ satisfying:

1. if $(x, y) \in \theta$, then $\mathrm{wt}(x) = \mathrm{wt}(y)$, $\ddot{\varepsilon}_i(x) = \ddot{\varepsilon}_i(y)$ and $\ddot{\varphi}_i(x) = \ddot{\varphi}_i(y)$;
2. if $(x, y) \in \theta$ and $\ddot{e}_i(x) \in M$, then $(\ddot{e}_i(x), \ddot{e}_i(y)) \in \theta$;
3. if $(x, y) \in \theta$ and $\ddot{f}_i(x) \in M$, then $(\ddot{f}_i(x), \ddot{f}_i(y)) \in \theta$;
4. if $(x_1, y_1), (x_2, y_2) \in \theta$, then $(x_1x_2, y_1y_2) \in \theta$.

Theorem 18 (Theorems 8.15, 8.19 and 8.20, pp. 116–119).

1. *The quasi-crystal monoid congruences on $\mathcal{M}$ form a lattice.*
2. *Given a surjective quasi-crystal monoid homomorphism $\psi : \mathcal{M} \to \mathcal{M}'$, then $\mathcal{M}' \cong \mathcal{M}/\ker\psi$.*
3. *If $\theta \subseteq \sigma$, then $(\mathcal{M}/\theta)/(\sigma/\theta) \cong \mathcal{M}/\sigma$.*

The general hypoplactic monoid

Definition 19 (Definition 8.21, p. 119).

Let $\mathcal{Q}$ be a seminormal quasi-crystal. The **hypoplactic congruence** on $\mathcal{Q}^{\ddot{*}}$ is a relation $\ddot{\sim}$ on Q^*, where $u \ddot{\sim} v$ if and only if

1. there exists a quasi-crystal isomorphism $\psi : \mathcal{Q}^{\ddot{*}}(u) \to \mathcal{Q}^{\ddot{*}}(v)$,
2. $\psi(u) = v$.

Theorem 20 (Theorem 8.23, p. 120).

$\ddot{\sim}$ is a quasi-crystal monoid congruence on $\mathcal{Q}^{\ddot{*}}$.

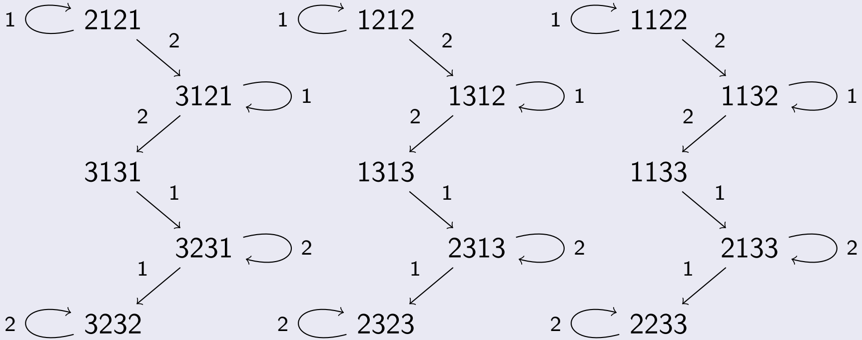
Definition 21 (Definition 8.24, p. 121).

The **hypoplactic quasi-crystal monoid**, or simply the **hypoplactic monoid**, associated to $\mathcal{Q}$ is $\mathrm{hypo}(\mathcal{Q}) = \mathcal{Q}^{\ddot{*}}/\ddot{\sim}$.

The general hypoplactic monoid

Example 5.

These components of the free quasi-crystal monoid $\mathcal{A}_3^{\ddot{*}}$ are isomorphic.



Therefore, in $\text{hypo}(\mathcal{A}_n)$ we have that $2121 = 1212 = 1122$.

The general hypoplactic monoid

Theorem 22 (Theorems 8.25 and 8.26, pp. 121–122).

Let $\mathcal{Q}$ be a seminormal quasi-crystal.

1. *If $u, v \in Q^*$ are such that uv is an isolated element of $\mathcal{Q}^{\ddot{*}}$, then*
$$uvw \ddot{\sim} uwv \ddot{\sim} wuv,$$
for any $w \in Q^*$.
2. *If $w \in Q^*$ is such that $wu \ddot{\sim} uw$, for any $u \in Q^*$, then w is an isolated element of $\mathcal{Q}^{\ddot{*}}$.*

Theorem 23 (Theorem 8.29, p. 124).

$\mathrm{hypo}(\mathcal{A}_n)$ *is anti-isomorphic to the classical hypoplactic monoid of rank n.*

Presentation for hypo($\mathcal{C}_2$)

Theorem 24 (Theorem 9.19, p. 136).

hypo($\mathcal{C}_2$) *has no finite presentation.*

Theorem 25 (Theorem 9.21, p. 138).

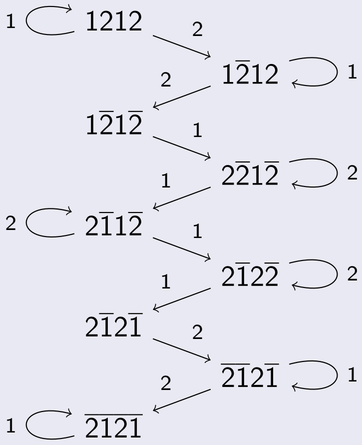
Any connected component of $\mathcal{C}_2^{\ddot{*}}$ is quasi-crystal isomorphic to one and only one of the following

1. $\mathcal{C}_2^{\ddot{*}}(1^m)$, $m \geq 0$;
2. $\mathcal{C}_2^{\ddot{*}}(2^{m_1}1^{m_2+1}2^{m_3+1}1^{m_4})$, $m_1, m_2, m_3, m_4 \geq 0$;
3. $\mathcal{C}_2^{\ddot{*}}(1^{m_1+1}2^{m_2}\overline{1}^{m_3+1})$, $m_1, m_2, m_3 \geq 0$ *with either* $m_1 = 0$ *or* $m_3 = 0$;
4. $\mathcal{C}_2^{\ddot{*}}(1^{m_1+1}2^{m_2+1}\overline{2}^{m_3+1}\overline{1}^{m_4+1})$, $m_1, m_2, m_3, m_4 \geq 0$ *with either* $m_1 = 0$ *or* $m_4 = 0$, *and either* $m_2 = 0$ *or* $m_3 = 0$.

Therefore, the elements in these connected components form a minimal set of representatives for the hypoplactic congruence $\ddot{\sim}$ on $\mathcal{C}_2^{\ddot{*}}$.

Example 6.

These components of the free quasi-crystal monoid $\mathcal{C}_2^{\ddot{*}}$ are isomorphic.



Theorem 26 (Theorem 9.23, p. 140).

Let X be a finite alphabet, and let $u, v \in X^*$. If hypo($\mathcal{C}_2$) *satisfies the identity $u = v$, then the following conditions are satisfied.*

1. $|u|_x = |v|_x$, *for all $x \in X$.*
2. *Until the first occurrence of a letter $x \in X$ in u and v, each letter of X occurs exactly the same number of times in u and v.*
3. *After the last occurrence of a letter $x \in X$ in u and v, each letter of X occurs exactly the same number of times in u and v.*

Theorem 27 (Theorem 9.25, p. 141).

The hypoplactic monoid hypo($\mathcal{C}_2$) *satisfies the identity*

$$xyxyxy = xyyxxy.$$

hypo($\mathcal{C}_n$)

Theorem 28 (Theorem 9.43, p. 153).

For $x, y, z \in C_n$ with $x \neq z$, we have that

1. $yzx \ddot{\sim} yxz$ *if and only if* $(y, x) = (1, \overline{1})$ *or* $(y, z) = (1, \overline{1})$;
2. $xzy \ddot{\sim} zxy$ *if and only if* $(x, y) = (1, \overline{1})$ *or* $(z, y) = (1, \overline{1})$;

Corollary 29 (Corollary 9.42, p. 153).

1. *For* $m, n \geq 2$, *there exists no injective homomorphism* ψ *from* $\mathrm{hypo}(\mathcal{A}_m)$ *to* $\mathrm{hypo}(\mathcal{C}_n)$ *such that* $\psi(x), \psi(y) \in C_n$ *for some* $x, y \in A_m$.
2. *For* $m \geq 3$ *and* $n \geq 2$, *no injective map from* A_m *to* C_n *can be extended to a homomorphism from* $\mathrm{hypo}(\mathcal{A}_m)$ *to* $\mathrm{hypo}(\mathcal{C}_n)$.

Theorem 30.

For $n \geq 3$, the set $\{1, 2\}$ is free on $\mathrm{hypo}(\mathcal{C}_n)$. *Therefore,* $\mathrm{hypo}(\mathcal{C}_n)$ *does not satisfy non-trivial identities.*

hypo($\mathcal{C}_n$)

Theorem 31.

Consider ψ to be the monoid homomorphism from A_{n-1}^* to C_n^* such that

$$\psi(x) = xn\bar{n}n\bar{n},$$

for each $x \in \{1, \dots, n-1\}$. Then, ψ induces an injective monoid homomorphism from hypo($\mathcal{A}_{n-1}$) *to* hypo($\mathcal{C}_n$).

Theorem 32.

Consider ψ to be the monoid homomorphism from C_{n-1}^* to C_n^* such that

$$\psi(x) = (x+1)1\bar{1} \qquad \text{and} \qquad \psi(\bar{x}) = (\overline{x+1})1\bar{1},$$

for each $x \in \{1, \dots, n-1\}$. Then, ψ induces an injective monoid homomorphism from hypo($\mathcal{C}_{n-1}$) *to* hypo($\mathcal{C}_n$).

References I

The results in this presentation report to

Guilherme, R. J. P. (2022). "Algebraic Theory of Quasi-crystals: A Generalization of the Hypoplactic Monoid and a Littelmann Path Model". PhD thesis. NOVA School of Science and Technology. URL: https://run.unl.pt/handle/10362/140587

supervised by António Malheiro and Alan J. Cain.

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