Looking for cores

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I - Motivation by representation theory

II - Cores and Lie theory

III - Atomic length in Weyl groups

I - Motivations

- <u>The Nakayama conjecture</u>

Theorem (Brauer-Robinson 1947)

$\lambda, \mu \vdash n$ lie in the same p-block of S_n $\quad \Leftrightarrow \quad$ λ, μ have the same p-core.

Example $\lambda = \ydiagram{6,4,1}$ and $\mu = \ydiagram{6,3,2}$ have same 3-core.

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Remark

- can define e-cores for $e \in \mathbb{Z}_{\geq 2}$
- e-cores are *e-regular* (no part is repeated e times or more)

- Enumerating e-cores

  - 2-cores are the triangular partitions $(r, r-1, \dots, 2, 1)$ for some r

$$\emptyset\,,\ \square\,,\ \ydiagram{2,1}\,,\ \ydiagram{3,2,1}\,,\ \dots$$

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Theorem (Granville-Ono 1996)

Let $e \geq 4$. For all $n \in \mathbb{N}$, there exists an e-core of size n.

Corollary Let $p \geq 5$. Every finite simple simple group has a defect 0 p-block.

- <u>Lattice structure on e-regular partitions</u>

For $i \in \mathbb{Z}/e\mathbb{Z}$, write $\lambda \xrightarrow{i} \mu$ if μ is obtained from λ by adding its *good i-box* (if it exists).

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- list all addable (A) and removable (R) $i-$boxes of λ in decreasing order,

Example $e = 3$ and

$$\lambda = \begin{array}{cccc} \square & \square & \boxed{2} & 0 \\ \square & \boxed{0} & 1 & \\ \square & 2 & & \\ \boxed{0} & & & \\ 2 & & & \end{array} \qquad \begin{array}{ll} i = 0 & ARR \\ i = 1 & A \\ i = 2 & RAA \end{array}$$

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- list all addable (A) and removable (R) $i-$boxes of λ in decreasing order,
- delete recursively all AR's,
- good box = leftmost remaining A.

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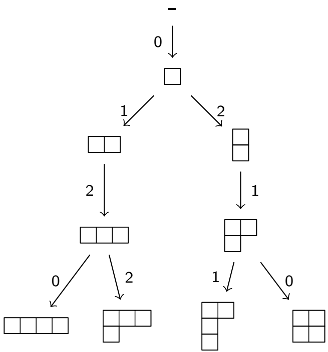
$i = 0$ $\cancel{AR}R$
$i = 1$ A
$i = 2$ RAA

Starting from the empty partition yields the *Kleshchev lattice*:



Example of $e = 3$.

The vertices are the e-regular partitions.

Remark

- $e = \infty \rightsquigarrow$ Young lattice
- gives the *modular branching rule* for S_n (Kleshchev 1995).

• Generalisations

Once we understand the story for S_n, it is natural to look at other situations such as:

1. Block theory for (unipotent) representations of finite classical groups (Fong-Srinivasan 1989).
2. Block theory for (cyclotomic) Hecke algebras (Lyle-Mathas 2007, Fayers, Jacon-Lecouvey, etc).

Today: focus on the second case.

II - Cores via crystals

$\mathfrak{g}$ (symmetrisable) Kac-Moody algebra $\rightsquigarrow$ classification in *Dynkin types*.

$\lambda \in P^+$ dominant weight $\rightsquigarrow$ $V(\lambda)$ irreducible highest weight $\mathfrak{g}$-module.

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Construction (Kashiwara 1990)

The structure of $V(\lambda)$ is controlled by its *crystal graph* $B(\lambda)$:

- vertices = *crystal* basis,
- arrows = action of the *crystal operators*.

It is the "combinatorial skeleton" of $V(\lambda)$.

Compatible with direct sums, tensor products, etc.

- Examples

1. $\mathfrak{g} = \mathfrak{sl}_e$, $V = \mathbb{C}^e = V(\omega_1) = V(\square)$

$$B = \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{e-1} \boxed{e}.$$

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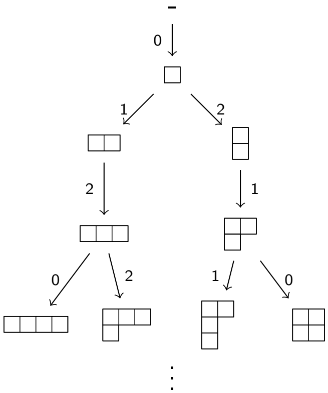
2. $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \omega_1 + \omega_2 = \yng(2,1)$.

$$\begin{tikzcd}
 & & \young(11,2) \arrow[dl, "1"'] \arrow[dr, "2"] & & \\
 & \young(12,2) \arrow[d, "2"'] & & \young(11,3) \arrow[d, "1"] & \\
B(\lambda) = & \young(13,2) \arrow[d, "2"'] & & \young(12,3) \arrow[d, "1"] & \hookrightarrow B^{\otimes 3} \\
 & \young(13,3) \arrow[dr, "1"'] & & \young(22,3) \arrow[dl, "2"] & \\
 & & \young(23,3) & &
\end{tikzcd}$$

3. $\mathfrak{g} = \widehat{\mathfrak{sl}_e}$ and $\lambda = \Lambda_0$.

Theorem (Misra-Miwa 1990)

The crystal $B(\Lambda_0)$ is given by the Kleshchev lattice.



- Action of the Weyl group

Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$.

Fix $i \in \mathbb{Z}/e\mathbb{Z}$. Removing all j-arrows, $j \neq i$, in the crystal yields a disjoint union of *i-strings*.

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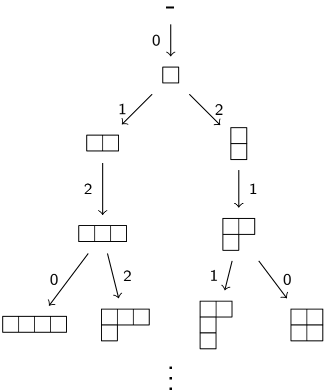
The generator s_i acts by reversing the i-strings:

$$\bullet \xrightarrow{i} b \xrightarrow{i} \bullet \xrightarrow{i} \cdots \xrightarrow{i} s_i(b) \xrightarrow{i} \bullet$$

Proposition

The orbit of the empty partition in $B(\Lambda_0)$ consists exactly of the e-cores.

Example Let $e = 3$.



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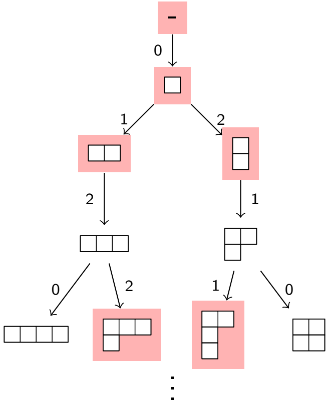
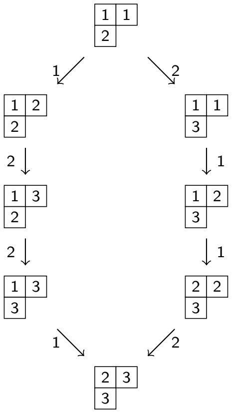


Theorem (Lascoux-Schützenberger 1990)

In finite type A, the orbit of the highest weight vertex consists of those tableaux $T = C_1 \cdots C_k$ such that $C_i \supseteq C_{i+1}$ for all $1 \leq i < k$.

Example Type A_2 and take $\lambda = \rho = \omega_1 + \omega_2 = \ydiagram{2,1}$.



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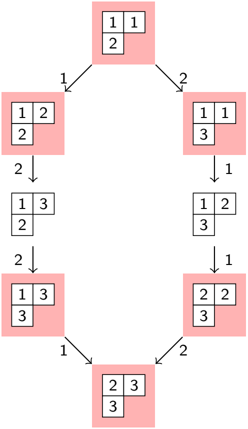

Generalisation of classical results by replacing

- cores $\leftarrow$ orbit of some highest weight vertex,
- size $\leftarrow$ depth in the crystal.

For instance, can we generalise Granville and Ono's result?

III - Atomic length in Weyl groups

- Inversion sets and (atomic) length

For $w \in W$, let

$$N(w) = \{\alpha \in \Phi_+ \mid w^{-1}(\alpha) \in \Phi_-\} = \text{inversion set of } w.$$

Recall that $|N(w)| = \ell(w)$.

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Definition

The *atomic length* of w is

$$L(w) = \sum_{\alpha \in N(w)} \mathrm{ht}(\alpha)$$

where $\mathrm{ht}(\alpha)$ is the number of simple roots needed to decompose α.

Example In type A_2, denote $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2$ the simple roots and $s_i = s_{\alpha_i} \in W$ the simple reflections.

w	$N(w)$	$\ell(w)$	$L(w)$
1	$\emptyset$	0	0
s_1	$\{\alpha_1\}$	1	1
s_2	$\{\alpha_2\}$	1	1
$s_1 s_2$	$\{\alpha_2, \alpha_1 + \alpha_2\}$	2	3
$s_2 s_1$	$\{\alpha_1, \alpha_1 + \alpha_2\}$	2	3
$s_1 s_2 s_1$	$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$	3	4

- Reformulation and generalisation

We have

$$L(w) = \langle \rho - w(\rho), \rho^\vee \rangle \quad \text{where } \rho = \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha.$$

So we define, for $\lambda \in P^+$

$$L_\lambda(w) = \langle \lambda - w(\lambda), \rho^\vee \rangle.$$

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Proposition

$L_\lambda(w)$ is the depth of $w(b_\lambda)$ in the crystal $B(\lambda)$, where b_λ denotes the highest weight vertex.

In particular:

- in type $A^{(1)}_{e-1}$, $L_{\Lambda_0}(w)$ is the *size* of the e-core partition $w(\emptyset)$.
- in type A_{e-1}, $L(w) = L_\rho(w)$ is the *entropy* of the permutation w.

- Finite Weyl groups

Let W be finite. Let e denote the rank of the corresponding root system. Clearly, L is maximal at w_0 (longest element).

Dynkin type	$L(w_0)$
A_e	$\frac{(e+1)e(e-1)}{6}$
B_e	$\frac{(e+1)e(4e-1)}{6}$
C_e	idem
D_e	$\frac{e(e-1)(2e-1)}{3}$

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Theorem (Chapelier-G. 2022)

Let $e \geq 3$. Then $L(W) = [\![0, L(w_0)]\!]$.

- Affine type A

Let $W = A^{(1)}_{e-1}$ and $\lambda \in P^+$ dominant weight of level ℓ.

- Crystal $B(\lambda)$ is realised by Uglov/Kleshchev/FLOTW ℓ-partitions.
- Highest weight vertex is $b_\lambda = (\emptyset, \dots, \emptyset)$.
- $L_\lambda(w) =$ size of the ℓ-partition $w(\emptyset, \dots, \emptyset)$.
- Simple characterisation of the orbit of $(\emptyset, \dots, \emptyset)$ using abaci (Jacon-Lecouvey 2021) similar to Lascoux and Schützenberger's.

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Questions

1. When is $L_\lambda : W \longrightarrow \mathbb{N}$ surjective?
2. (weaker) When is $\mathbb{N} \setminus L(W)$ finite?

- Case $\lambda = \Lambda_0$. Then L_{Λ_0} is surjective (Granville-Ono theorem).

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- General case:

$$\lambda = \overline{\lambda} + \ell\Lambda_0 \quad \text{and} \quad w = \overline{w}\,\mathrm{t}_\beta$$

where $\overline{\lambda} \in P^+_{fin}$, $\overline{w} \in W_{fin}$ and $\beta = \sum_{i=1}^n b_i\alpha_i$.

Theorem (Chapelier-G. 2022)

$$L_\lambda(w) = L_{\overline{\lambda}}(\overline{w}) + \ell L_{\Lambda_0}(w) + K(\beta \mid \overline{\lambda})$$

where K is a constant depending on the type.

For some other particular λ's, Question (2) has a positive answer (work in progress with E. Norton)...