

## Spanning trees in the graph of pairwise comparisons

Sándor Bozóki

(Institute for Computer Science and Control, Hungarian Academy of Sciences)

Pairwise comparisons are frequently applied in multi-criteria decision making and ranking. The set of known comparisons can be complete or incomplete. Vitaliy Tsyganok's works (since 2000) drew attention to the significance of the edge-minimal, connected sets, i.e., spanning trees in the graph of comparisons. Earlier works focused on particular spanning trees, such as the star graph.

The mini-course presents some, mostly recent results on the weight vectors calculated from sets of comparisons, corresponding to spanning trees.

The weight vector calculated from a complete or incomplete pairwise comparison matrix by the logarithmic least squares method, can also be written as the geometric mean of weight vectors, calculated from the comparisons associated to the spanning trees of the comparisons' graph. The Laplacian matrix plays the main role in the proof.

Furthermore, for any number  $N$  of edges, larger than the number of items minus two, the weight vector calculated from a complete pairwise comparison matrix by the logarithmic least squares method, can also be written as the geometric mean of weight vectors, calculated from the comparisons associated to the connected subgraphs with  $E$  edges. The proof is illustrated by a special symmetry in the GRAPH of connected labelled graphs (there is an EDGE between two graphs if one of them can be given by the addition of one edge to the other graph). Spanning trees are at the top, the complete graph is at the bottom of this multi-partite GRAPH. However, the proposition does not extend to the incomplete case due to a broken symmetry.

Simulations show that a star graph is optimal among (now unlabelled) spanning trees if we compare the corresponding weight vectors to the weight vector calculated from the complete pairwise comparison matrix. However, as we increase the number of edges, more and more regular graphs appear among the optimal graphs.

Pareto optimality (efficiency) of a weight vector is a natural requirement. Once we can find a *better* (dominating) weight vector, which estimates all the matrix elements at least as good, and strictly better in at least one position, it is hard to argue with the use of the dominated one. A weight vector calculated from a spanning tree's comparisons is always Pareto optimal.

Most results are presented within the framework on the multiplicative/additive/reciprocal pairwise comparison matrix format, used in the Analytic Hierarchy Process related literature. However the direct relation to physical realizations such as electric circuits is also highlighted.