Spanning trees
in the graph of pairwise comparisons

Sándor Bozóki

https://scholar.google.com/citations?hl=en&user=YSwEgM8AAAAJ&view_op=list_works&sortby=pubdate

The 13th Combinatorics Days

Covilhã, July 13-14, 2023

The aim of multiple criteria decision analysis

The aim is **to select the overall best one** from a finite set of *alternatives*, with respect to a finite set of **attributes (criteria)**,
or,
to rank the alternatives,
or,
to classify the alternatives.

Properties of multiple criteria decision problems

- criteria contradict each other
- there is not a single best solution, that is optimal with respect to each criterion
- subjective factors influence the decision
- contradictive individual opinions have to be aggregated

Examples of multi criteria decision problems

- tenders, public procurements, privatizations
- evaluation of applications
- environmental studies
- ranking, classification

Decomposition of the goal: tree of criteria

- main criterion 1
  - criterion 1.1
  - criterion 1.2
  - criterion 1.3
  - criterion 1.4
  - criterion 1.5
- main criterion 2
  - criterion 2.1
  - criterion 2.2
- main criterion 3
  - criterion 3.1
    - subcriterion 3.1.1
    - subcriterion 3.1.2
  - criterion 3.2

Estimating weights from pairwise comparisons

'How many times criterion 1 is more important than criterion 2?'

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

is given, where for any $i, j = 1, \dots, n$ indices

$$a_{ij} > 0, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

The aim is to find the $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top \in \mathbb{R}^n_+$ weight vector such that ratios $\frac{w_i}{w_j}$ are *close enough* to a_{ij}s.

Evaluation of the alternatives

Alternatives are evaluated directly, or by using a function, or by pairwise comparisons as before.

'How many times alternative 1 is better than alternative 2 with respect to criterion 1.1?'

$$\mathbf{B} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & 1 & b_{23} & \dots & b_{2m} \\ b_{31} & b_{32} & 1 & \dots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & 1 \end{pmatrix}$$

Weighting methods

Eigenvector Method (Saaty): $\mathbf{A}\mathbf{w} = \lambda_{max}\mathbf{w}$.

Logarithmic Least Squares Method (LLSM):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2$$

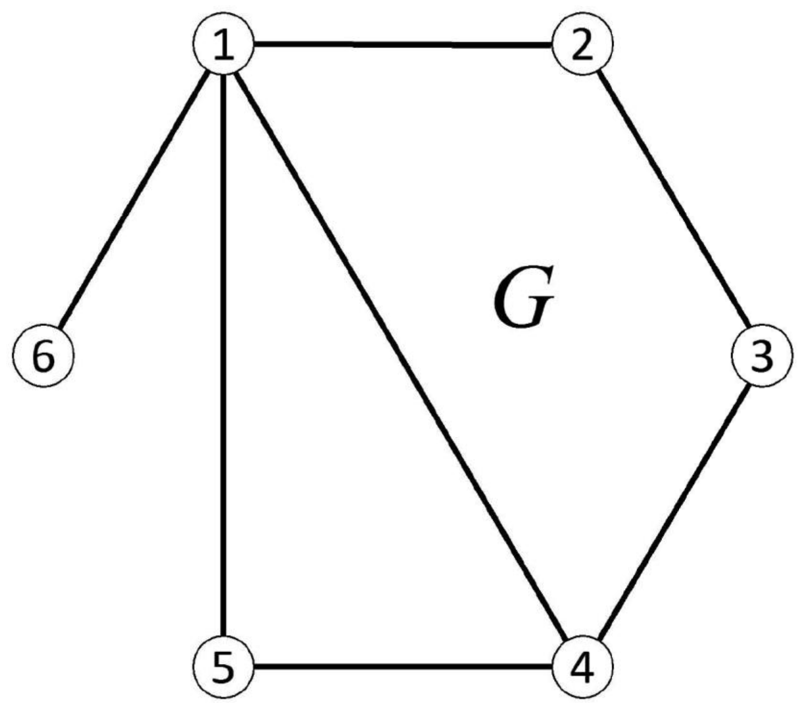
$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \quad i = 1, 2, \ldots, n.$$

incomplete pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

incomplete pairwise comparison matrix and its graph

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



The Logarithmic Least Squares (LLS) problem

$$\min \sum_{\substack{i,j: \\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

The most common normalizations are $\sum_{i=1}^{n} w_i = 1$, $\prod_{i=1}^{n} w_i = 1$ and $w_1 = 1$.

Theorem (Bozóki, Fülöp, Rónyai, 2010): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph G is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

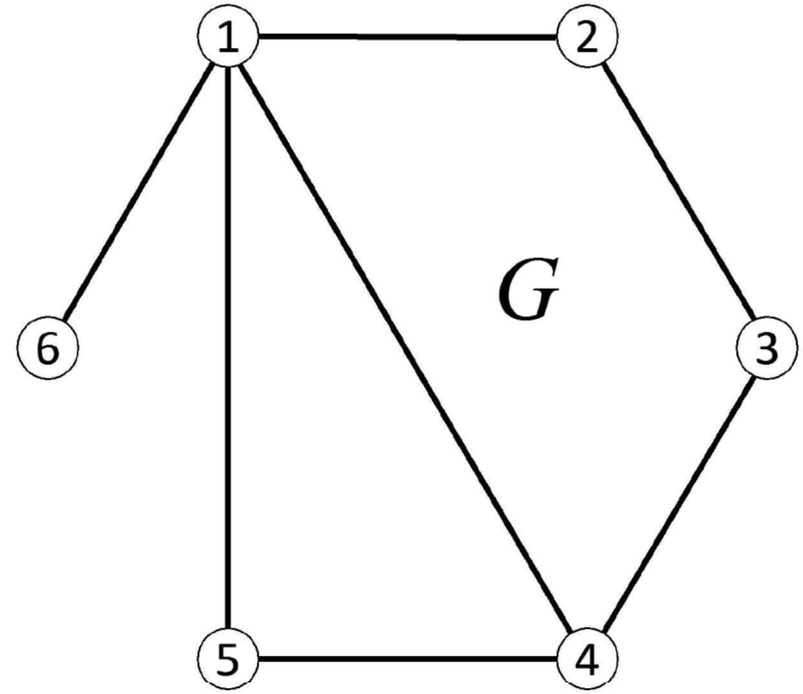
$$(\mathbf{L}\mathbf{y})_i = \sum_{k:e(i,k)\in E(G)} \log a_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

$$y_1 = 0$$

where $\mathbf{L}$ denotes the Laplacian matrix of G (ℓ_{ii} is the degree of node i and $\ell_{ij} = -1$ if nodes i and j are adjacent).

example

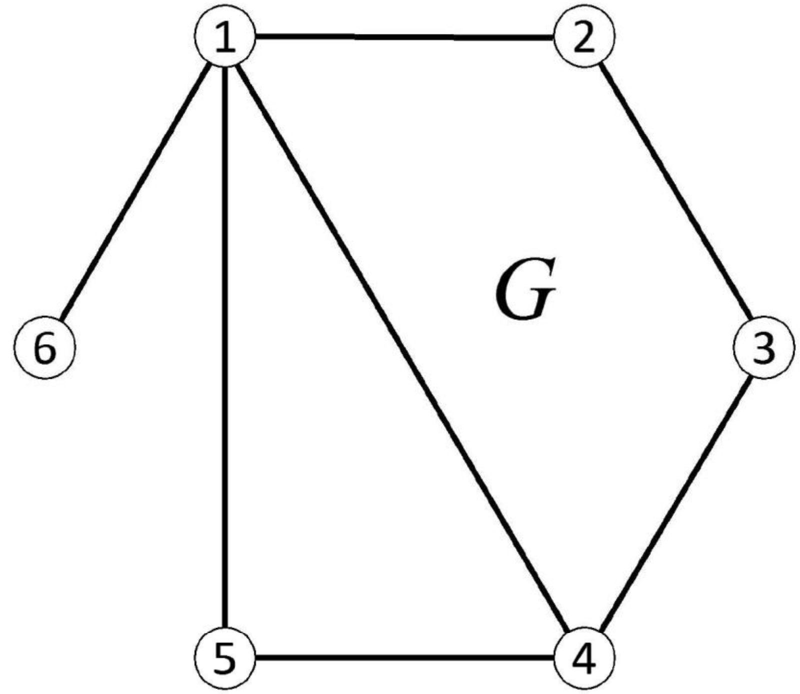
$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(=0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log(a_{12}\, a_{14}\, a_{15}\, a_{16}) \\ \log(a_{21}\, a_{23}) \\ \log(a_{32}\, a_{34}) \\ \log(a_{41}\, a_{43}\, a_{45}) \\ \log(a_{51}\, a_{54}) \\ \log a_{61} \end{pmatrix}$$

The spanning tree approach (Tsyganok, 2000, 2010)

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



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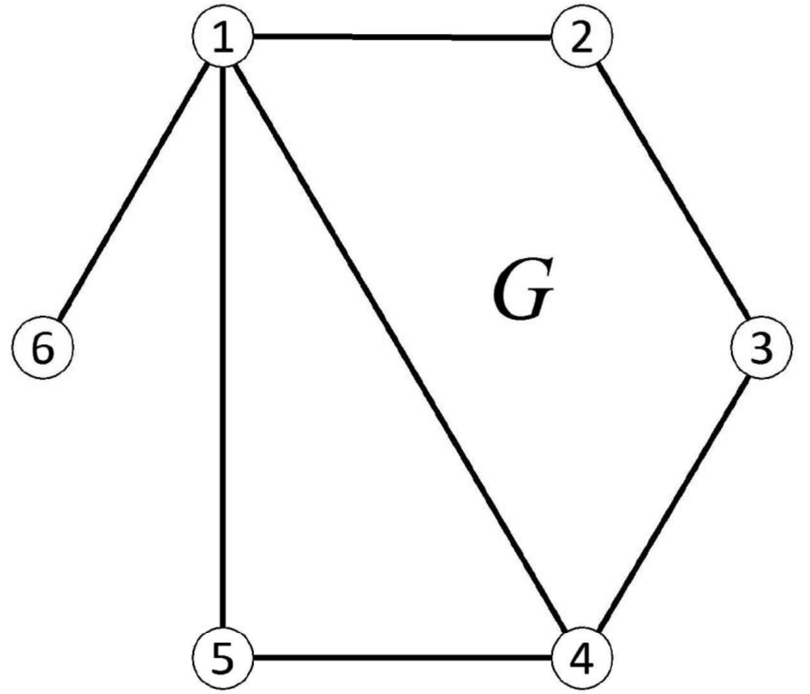

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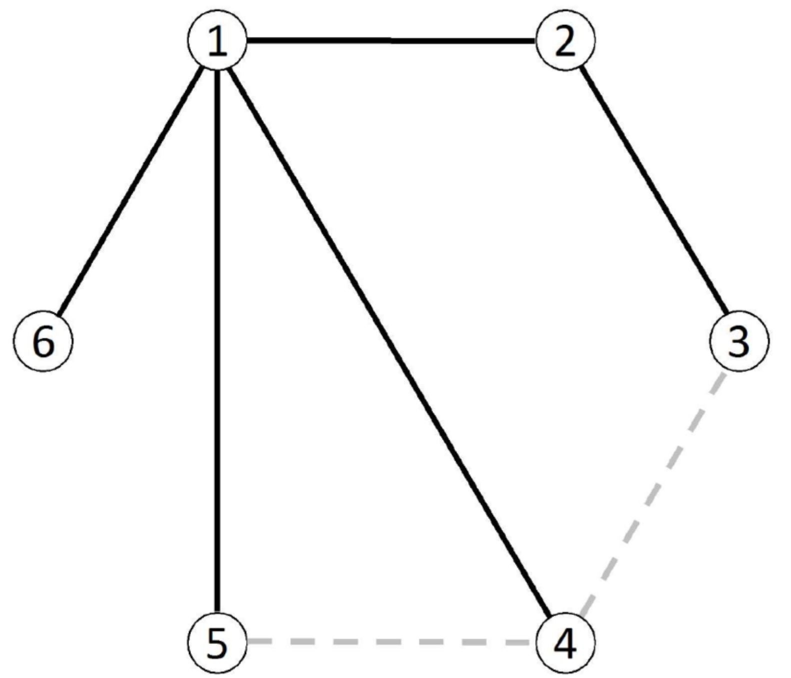

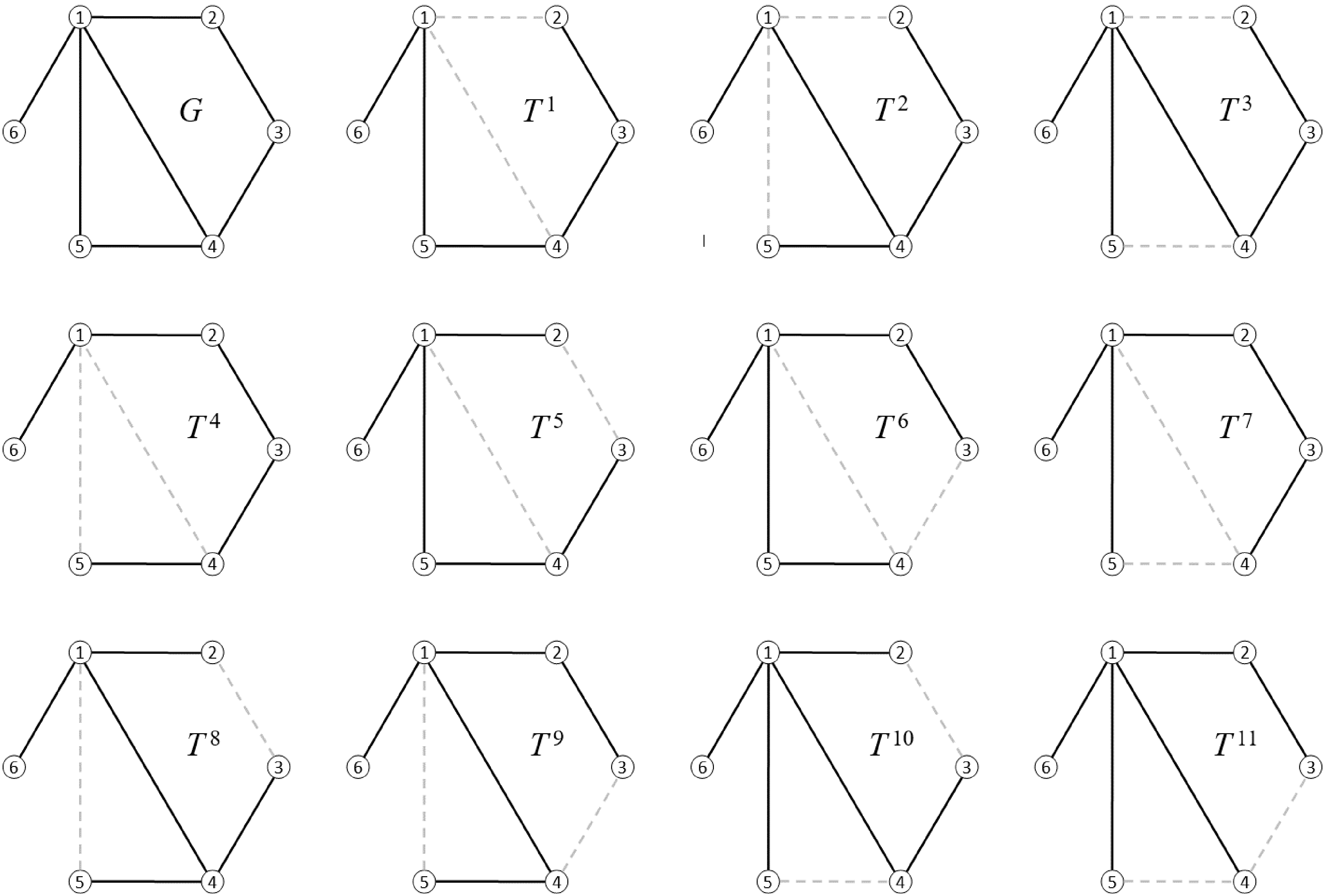
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The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.

Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

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Theorem (Bozóki, Tsyganok): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.

proof

Let G be the connected graph associated to the (in)complete pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes i and j is denoted by $e(i,j)$.

The Laplacian matrix of graph G is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G, where S denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in T^s.

Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree T^s. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $\mathbf{y}^s := \log \mathbf{w}^s,\ s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.

proof

Let $\mathbf{w}^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $\mathbf{y}^{LLS} := \log \mathbf{w}^{LLS}$, then

$$\left(\mathbf{L}\mathbf{y}^{LLS}\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

where $b_{ik} = \log a_{ik}$ for all $(i,k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $(i,k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$\left(\mathbf{L}\frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^{s}\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n.$$

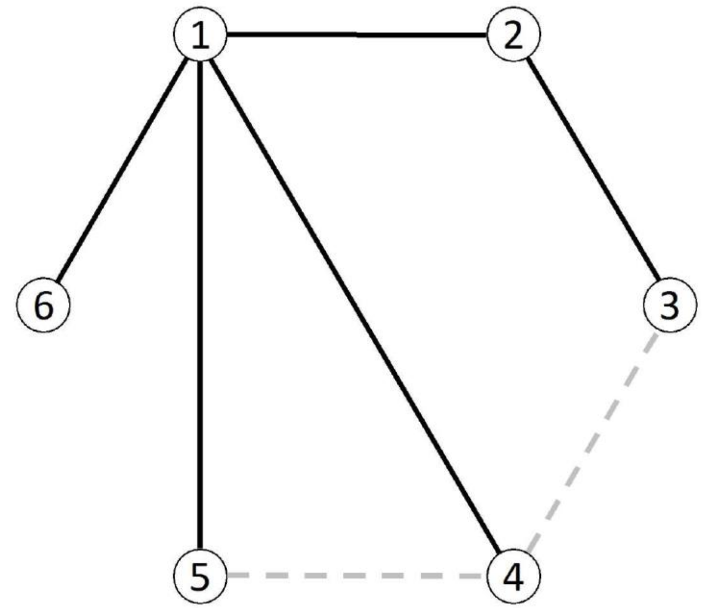
proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of G.

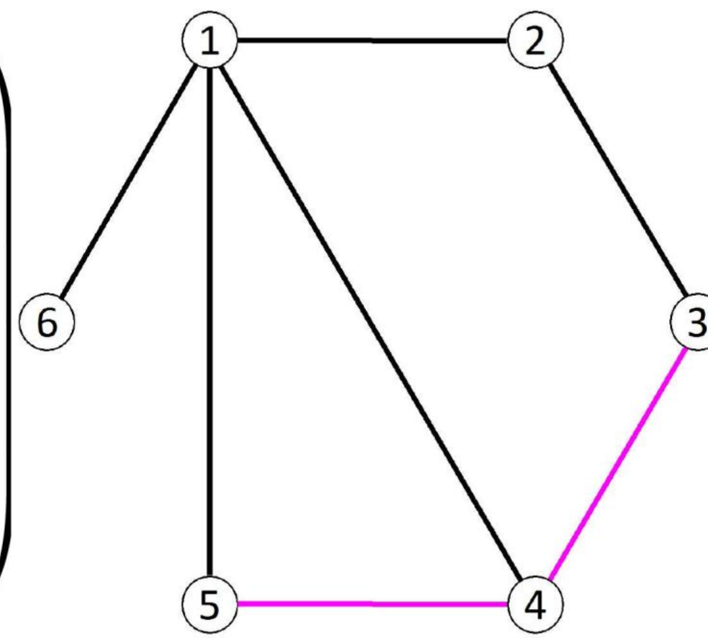
Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$.
Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

proof

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{32}a_{21}a_{14} & & \\ a_{41} & & a_{41}a_{12}a_{23} & 1 & a_{41}a_{15} & \\ a_{51} & & & a_{51}a_{14} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



proof

Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$. Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.

Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

Since weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree T^s, it is the optimal solution of the LLS problem regarding $\mathbf{A}^s$, too. Equivalently, the following system of linear equations holds.

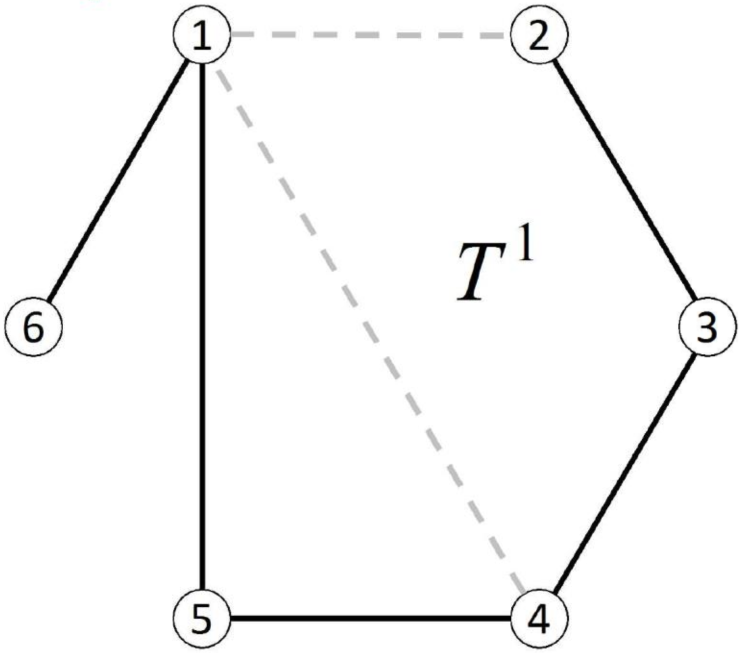
$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\backslash E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

proof

Lemma

$$\sum_{s=1}^{S}\left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^{s}\right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

proof of the lemma



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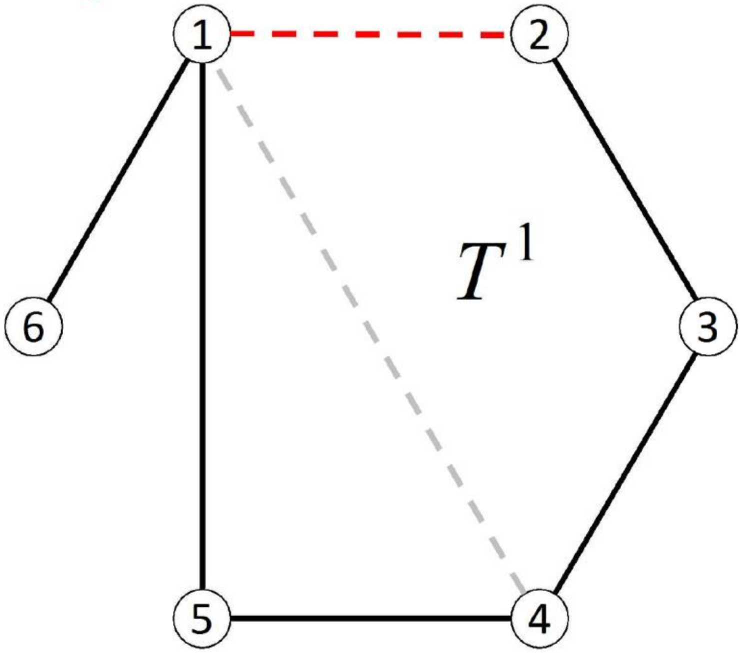

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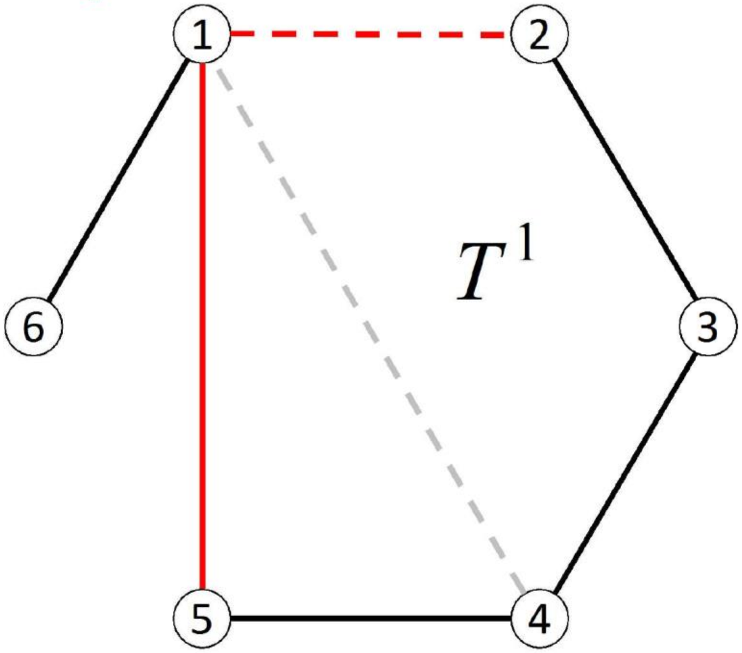

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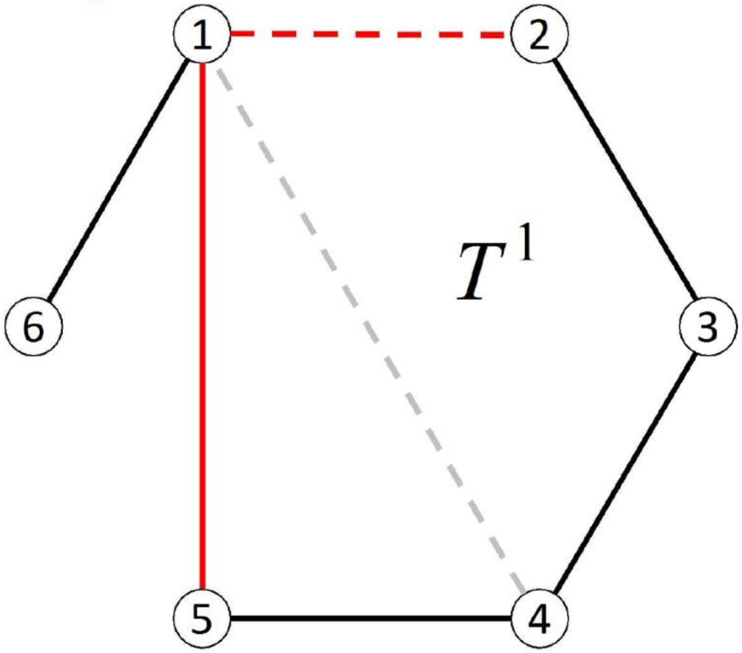
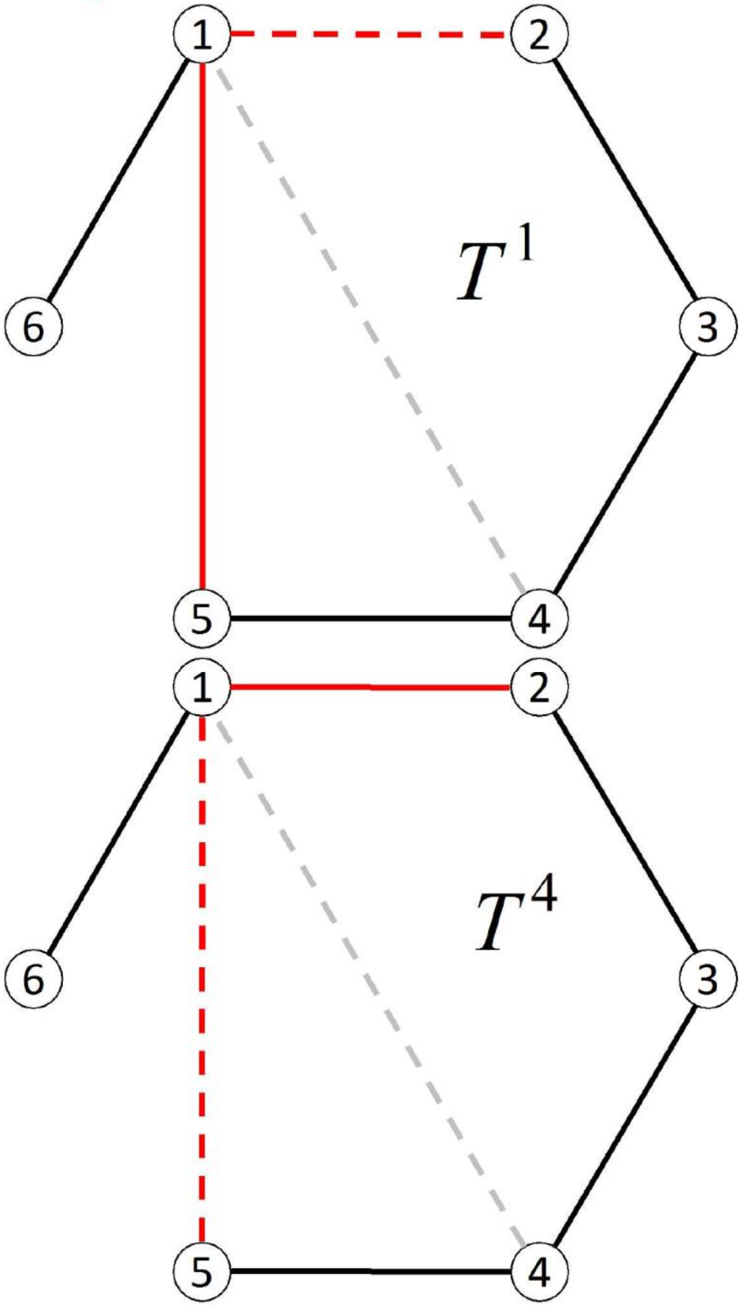


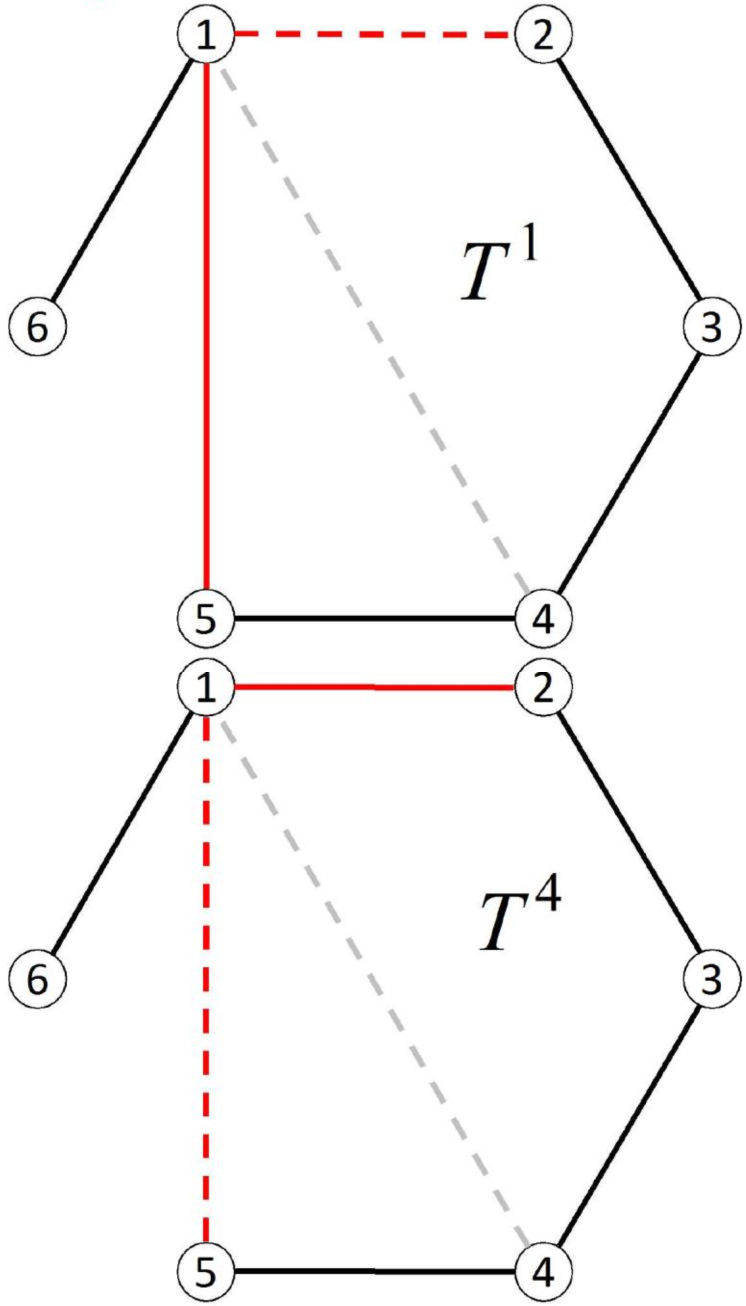
$$b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32}$$

proof of the lemma



$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

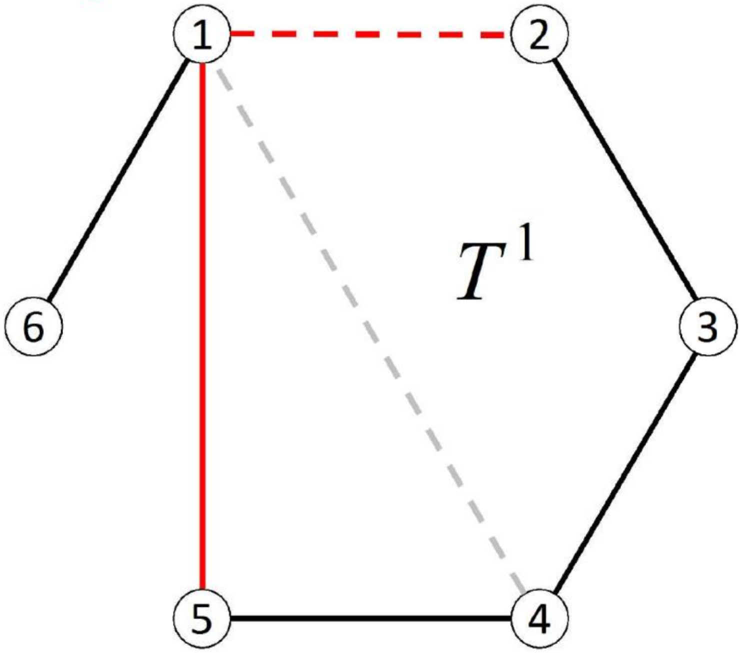
proof of the lemma



$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

proof of the lemma



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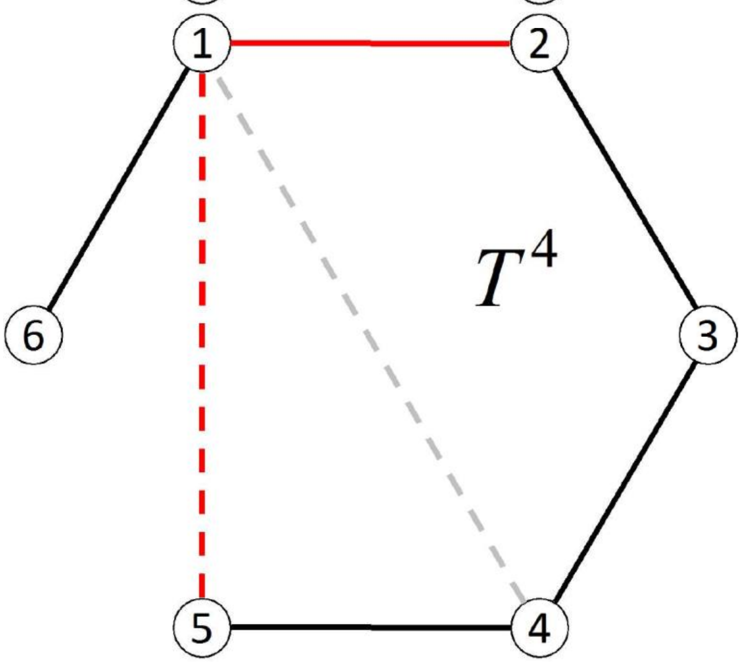

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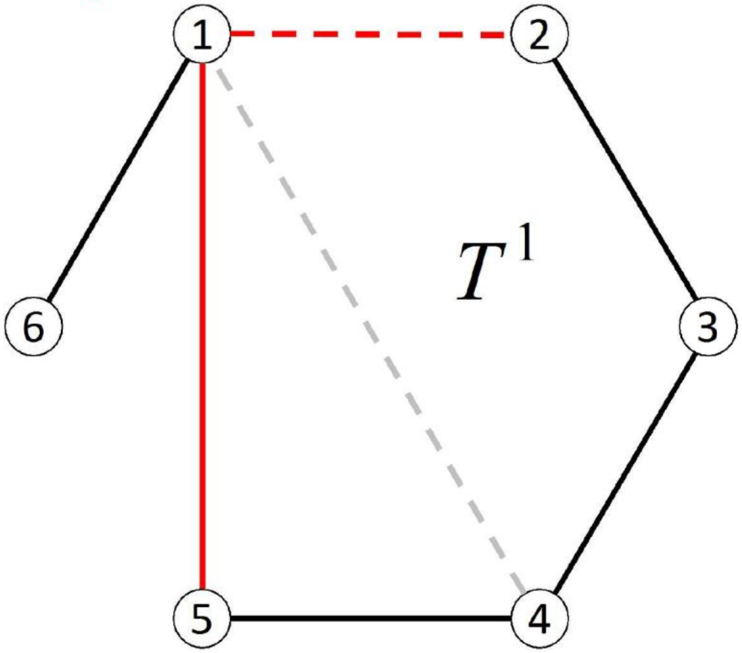

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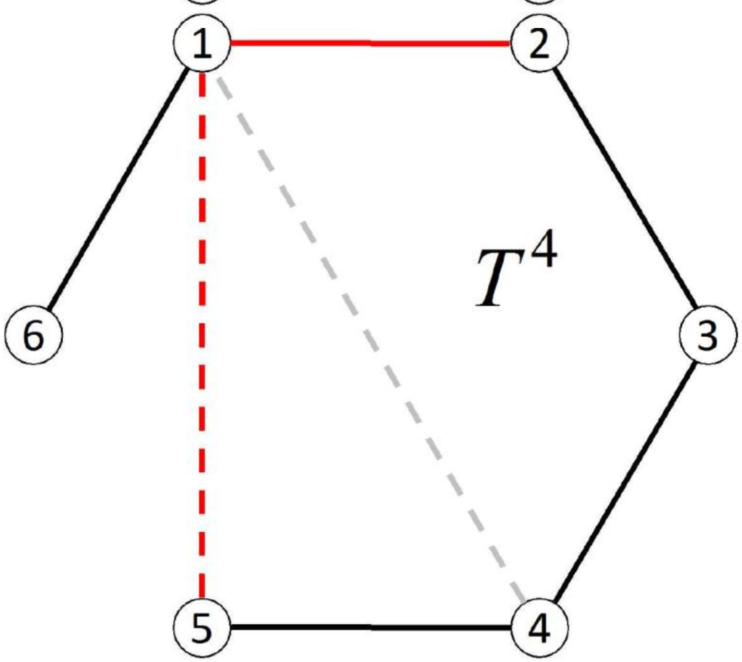
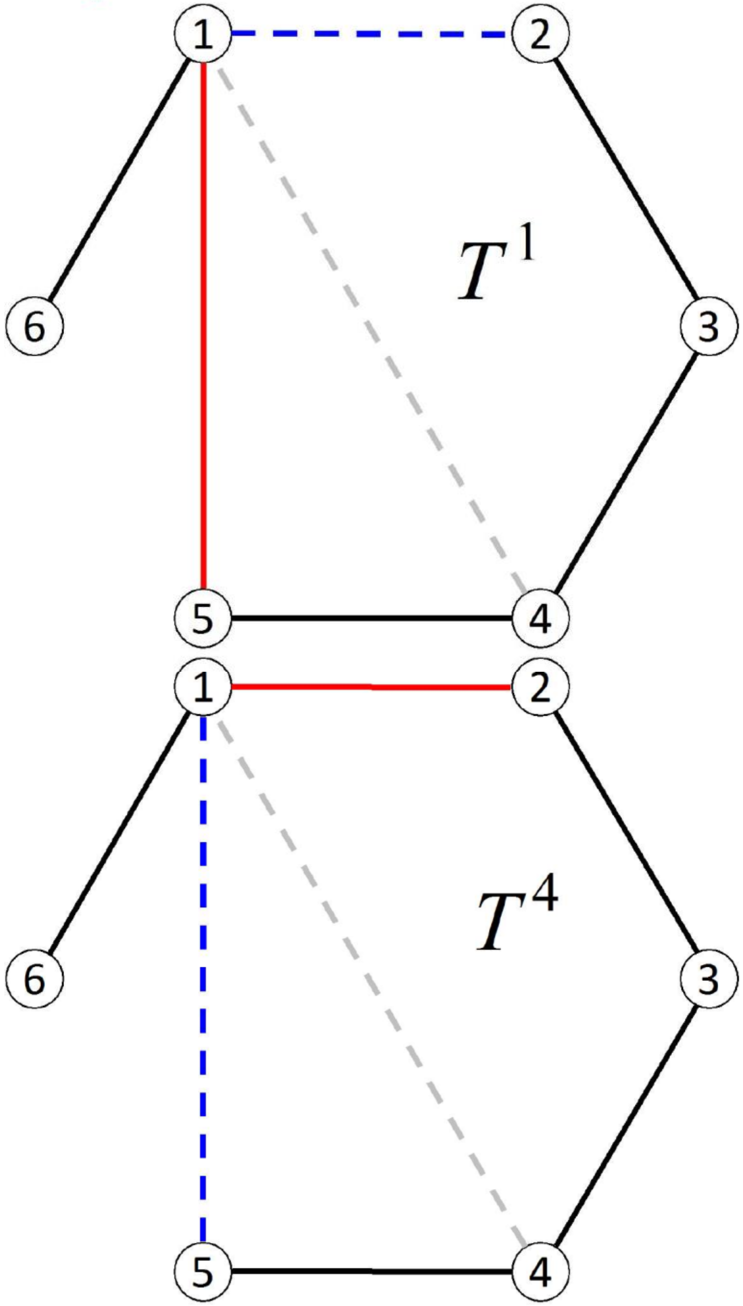


$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

proof of the lemma



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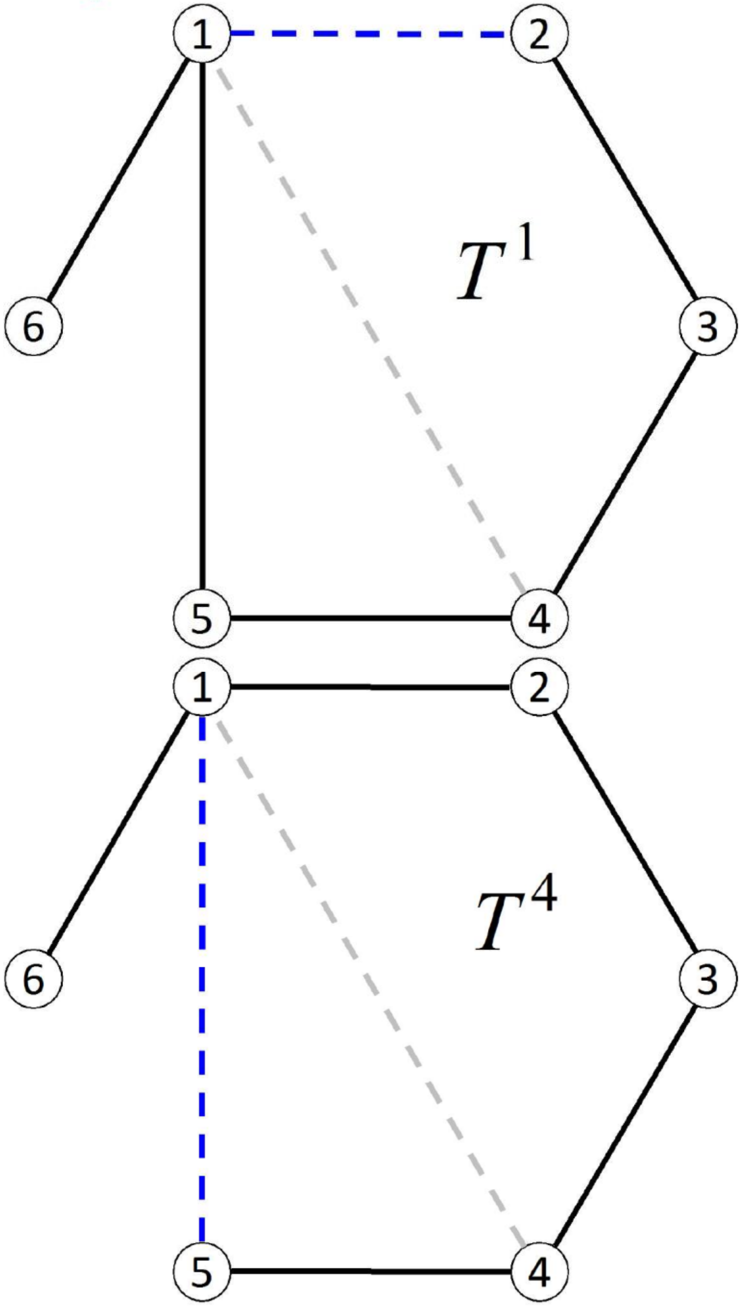
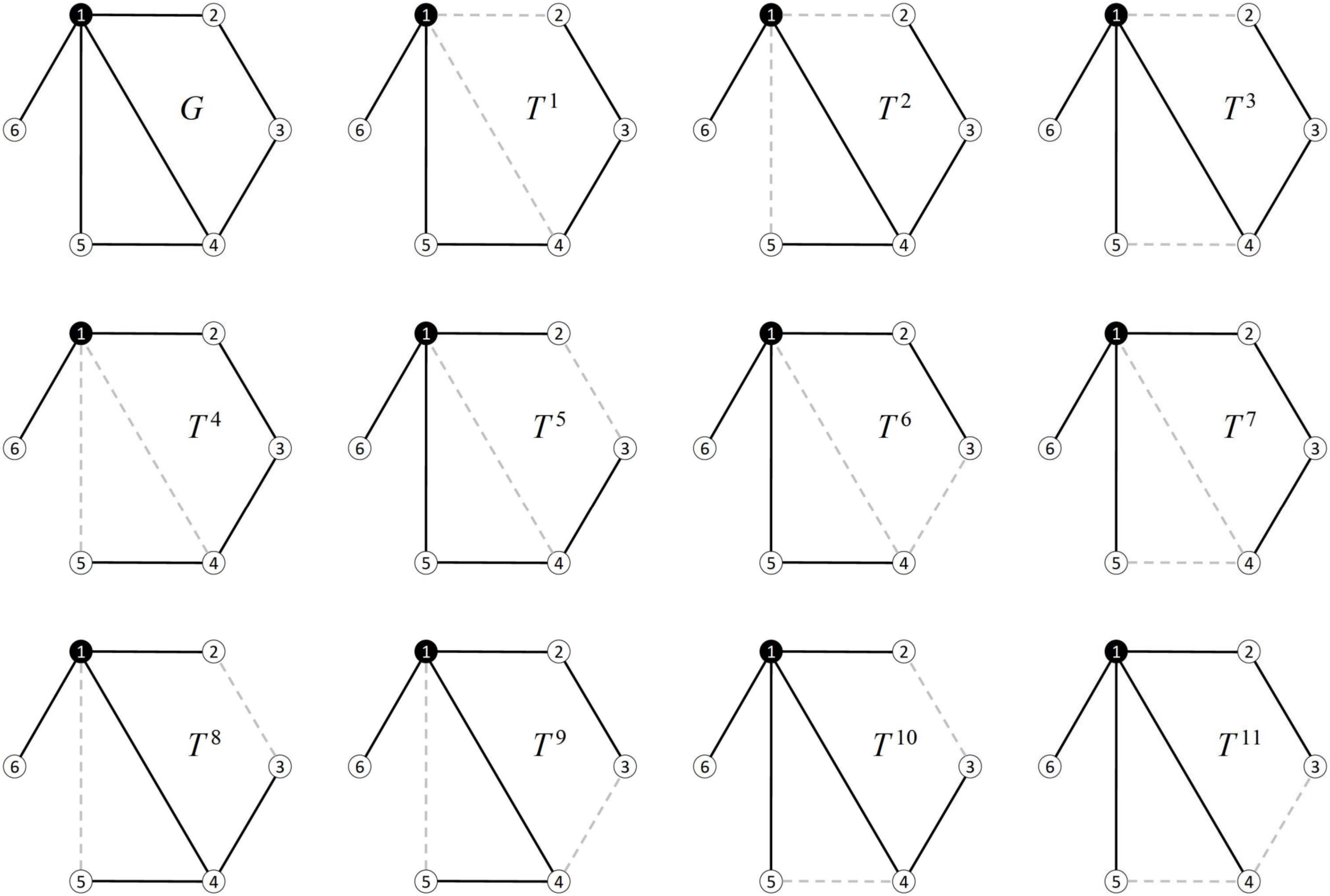


$$b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b_{15}^4 = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b_{12}^1 + b_{15}^4 = b_{12} + b_{15}$$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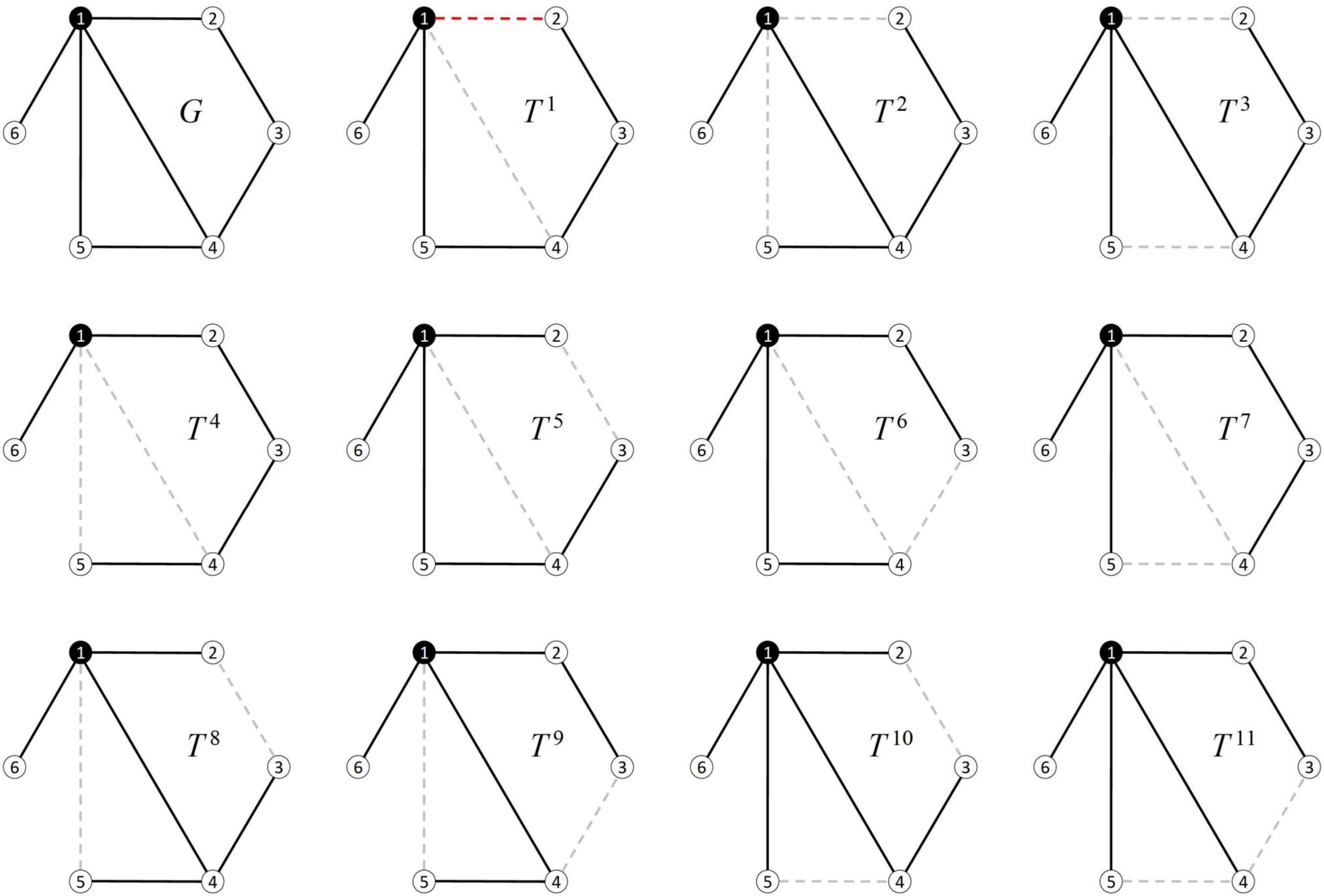

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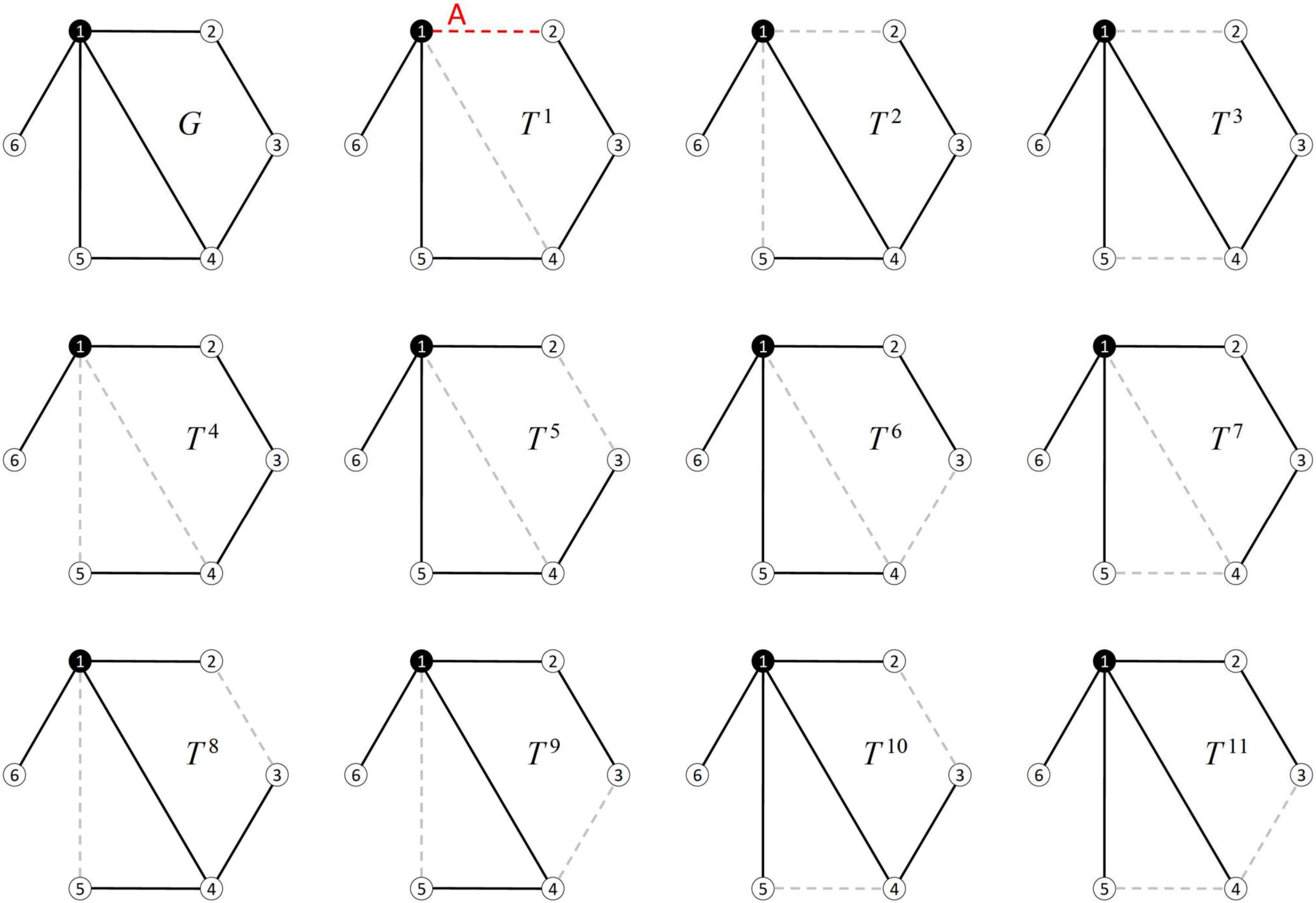

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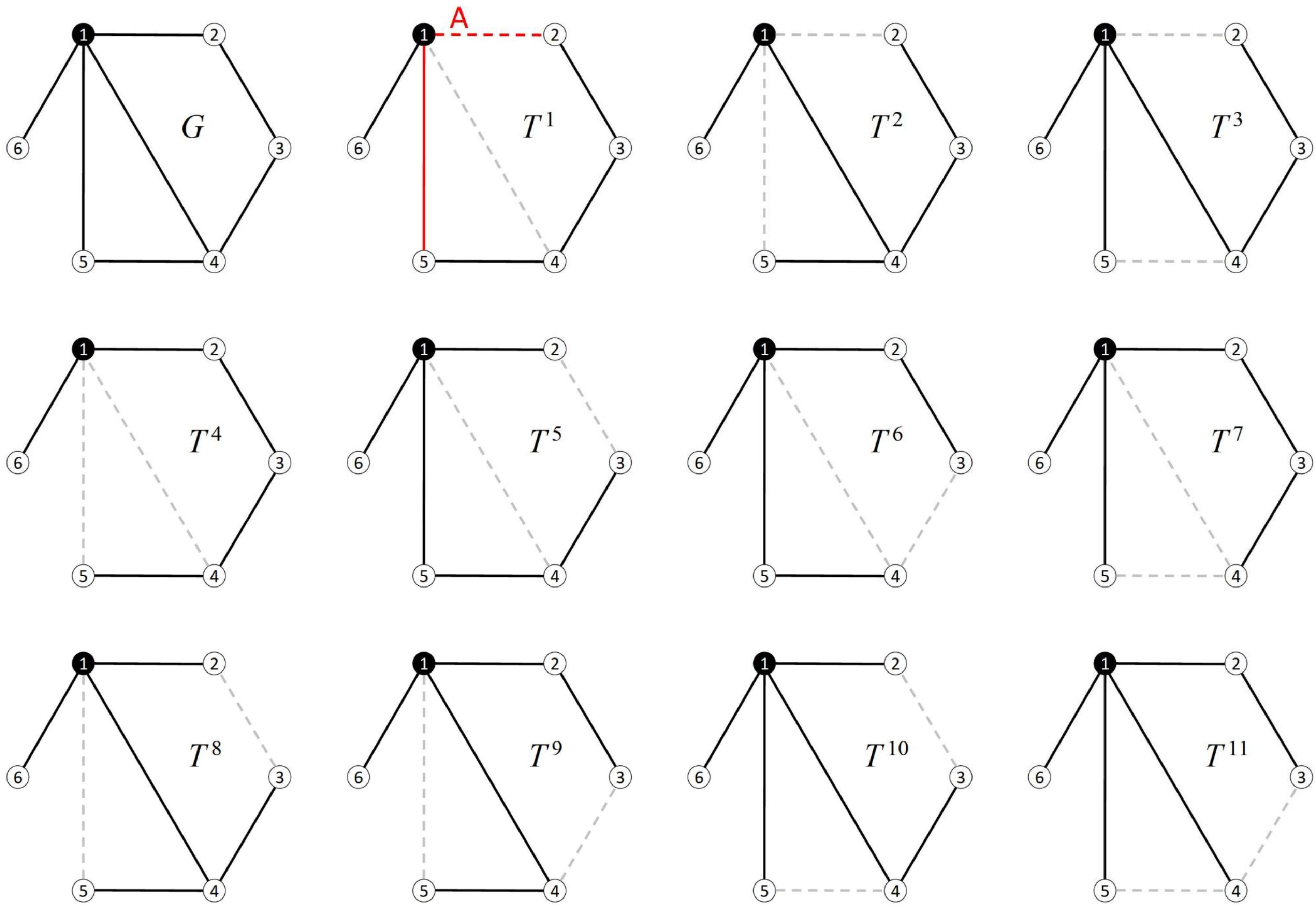
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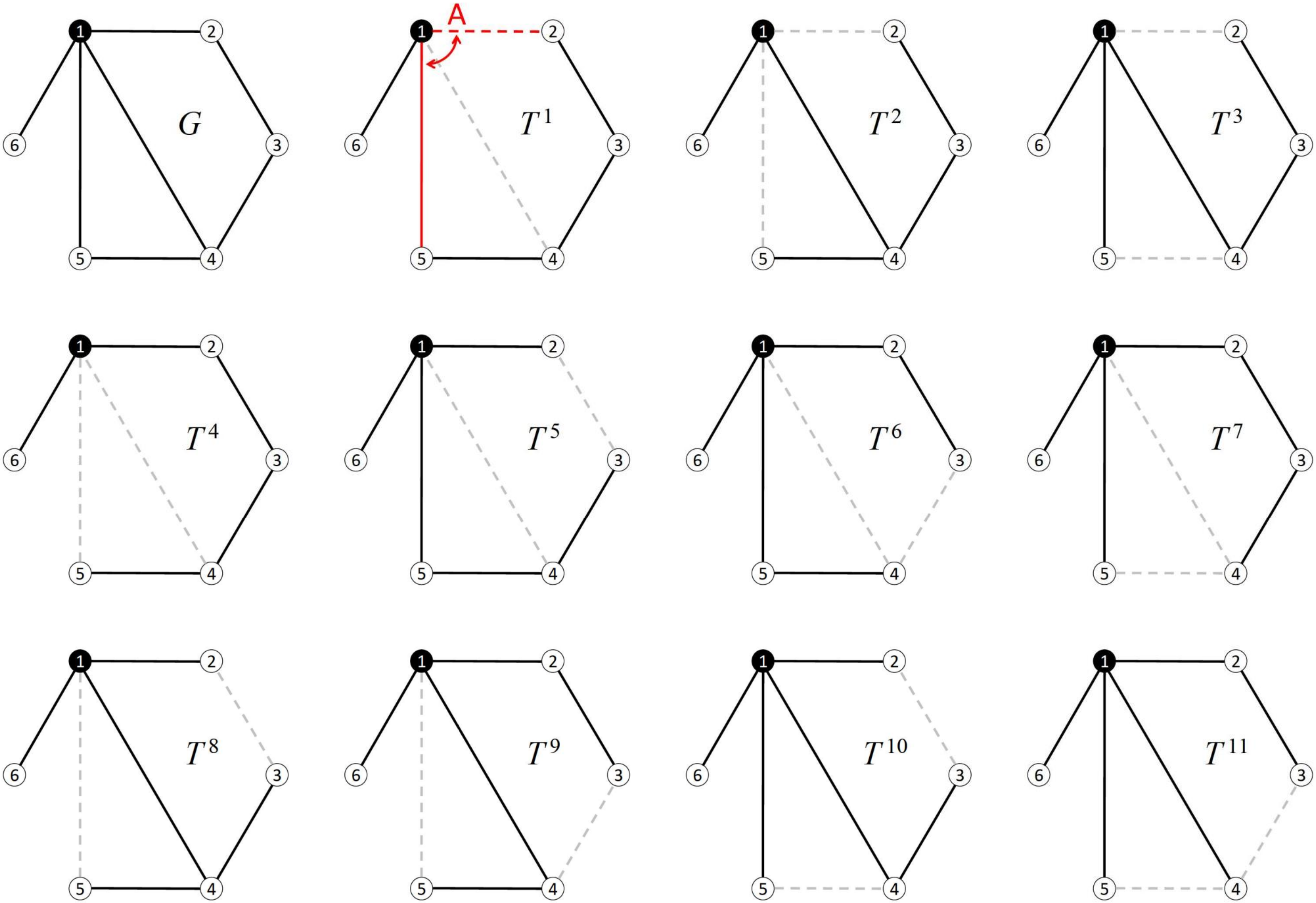

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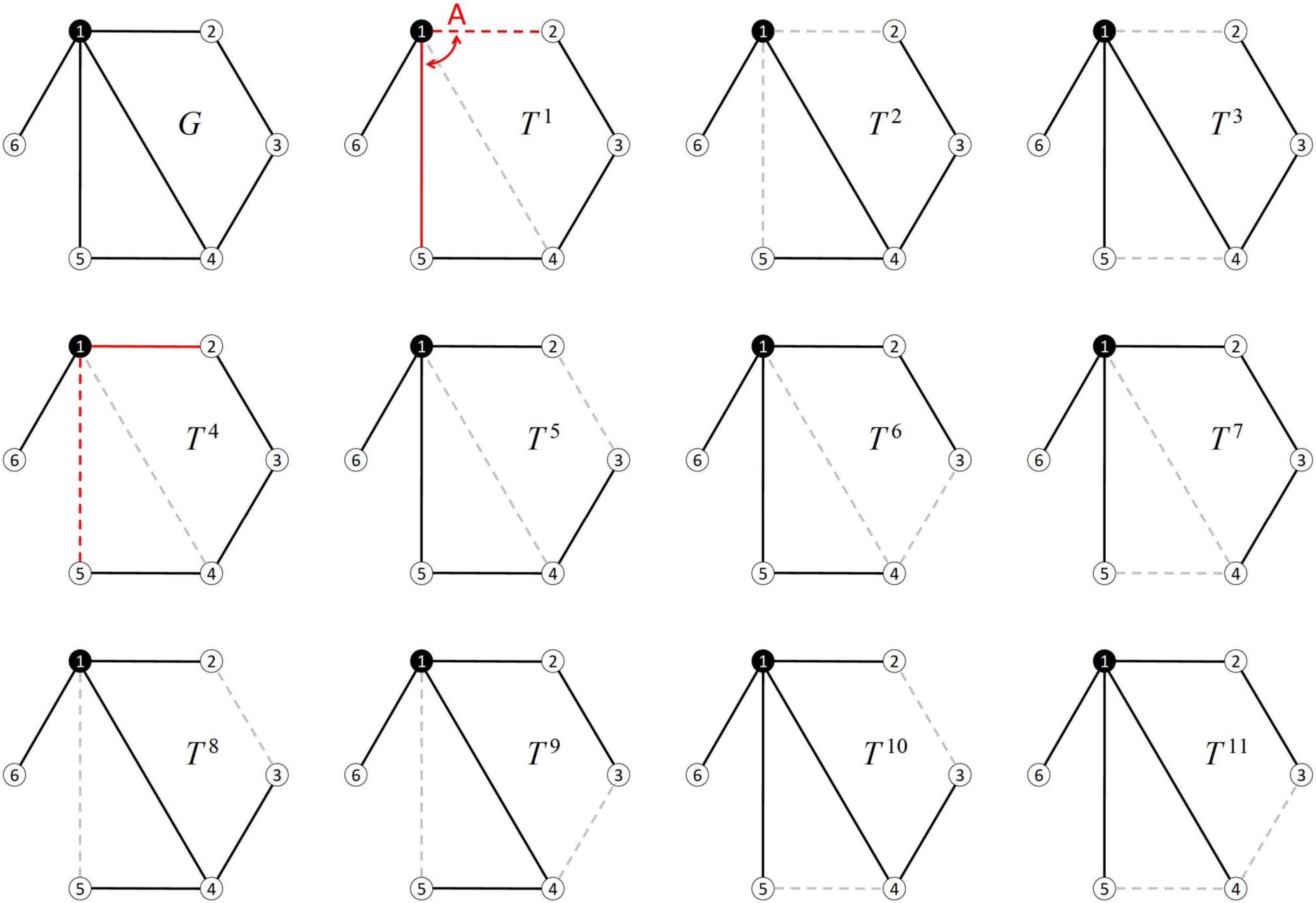

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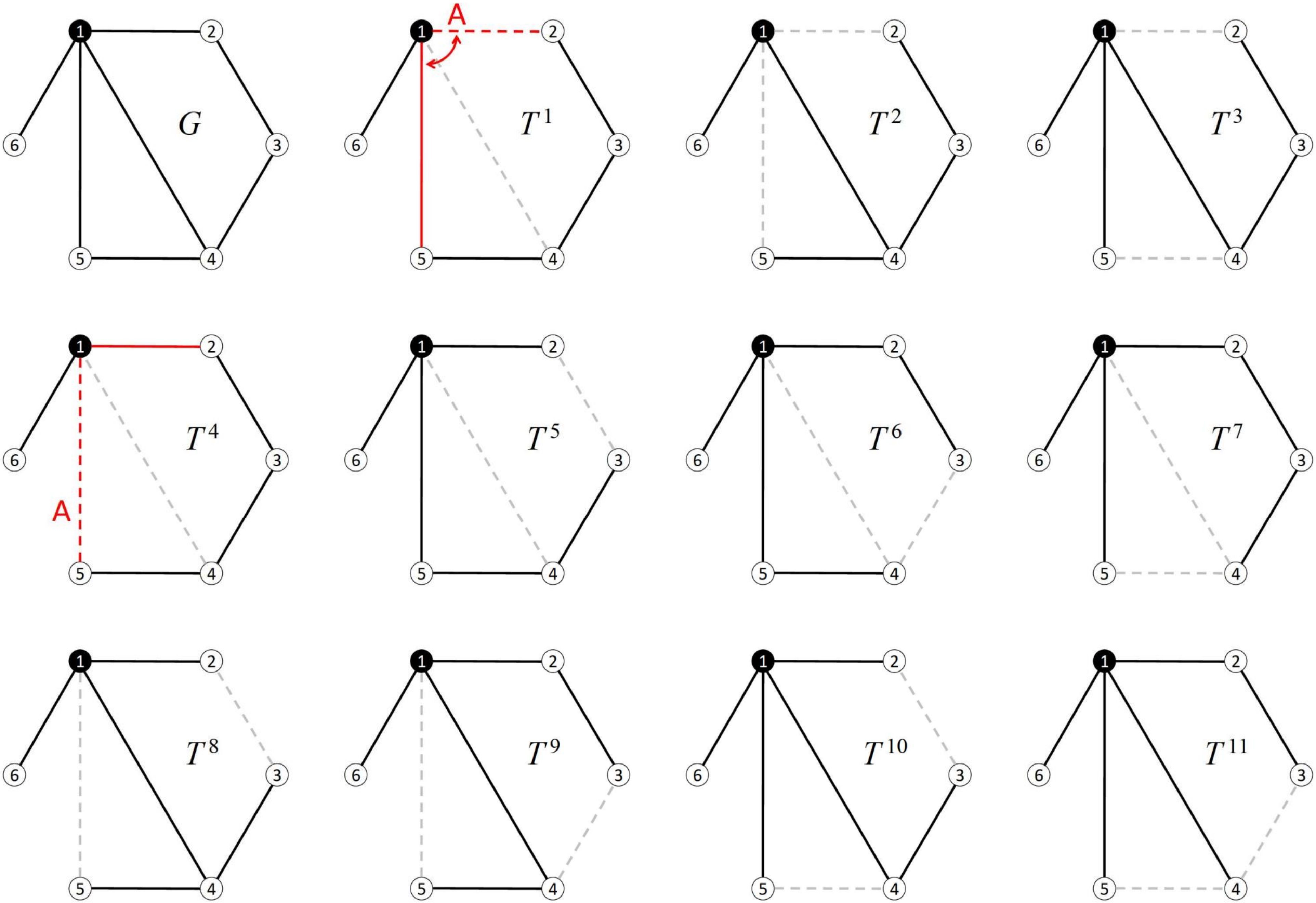
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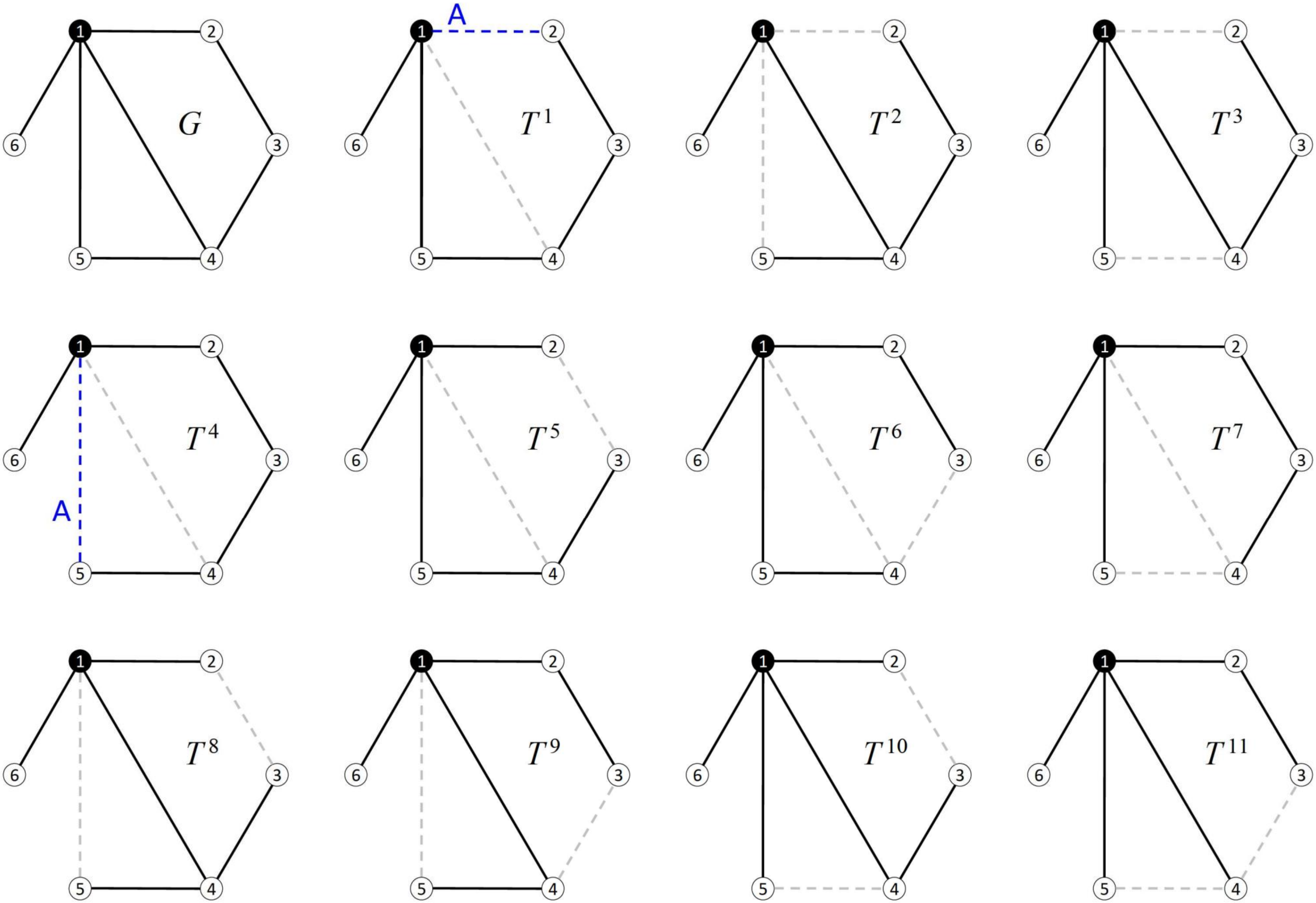
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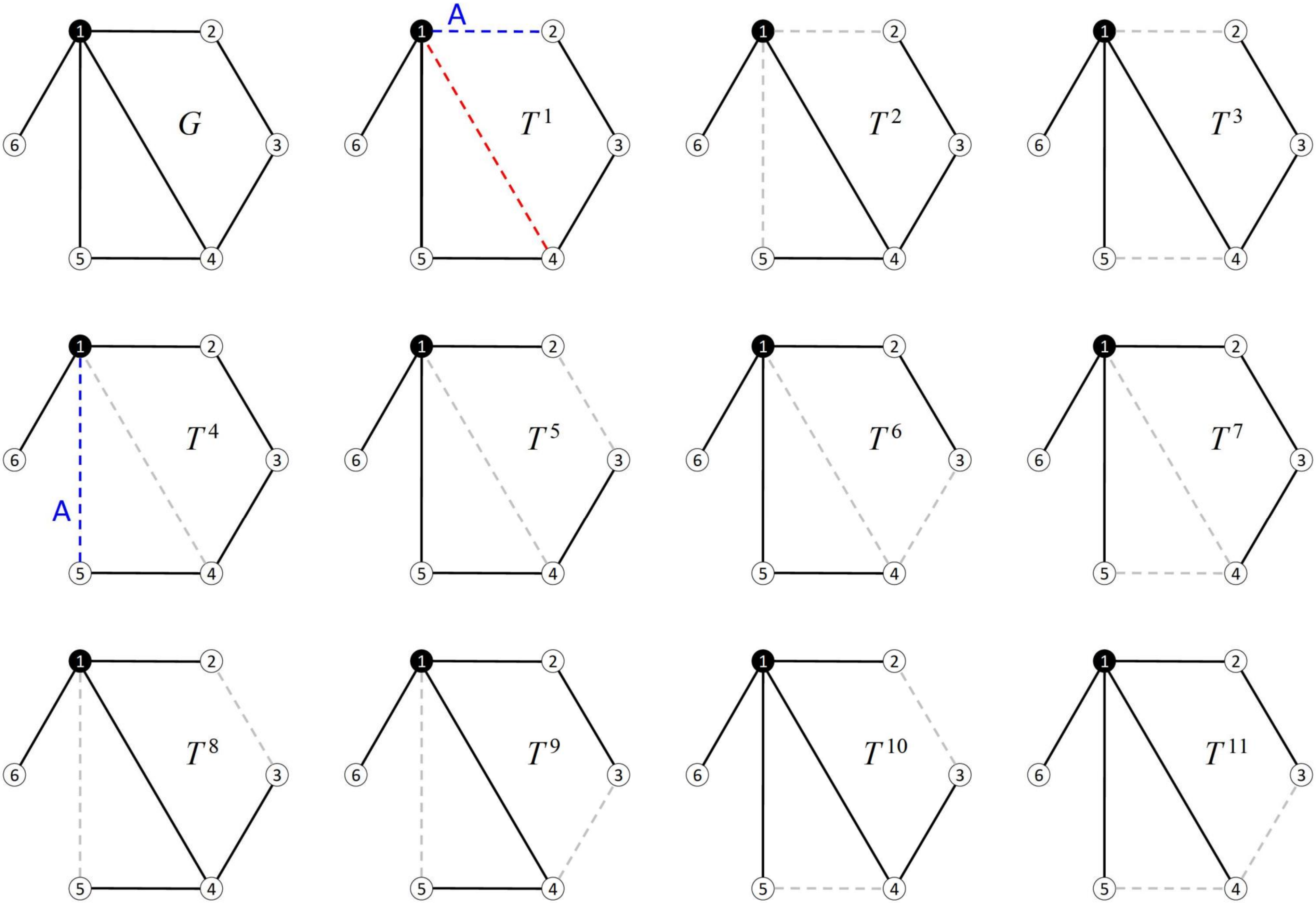

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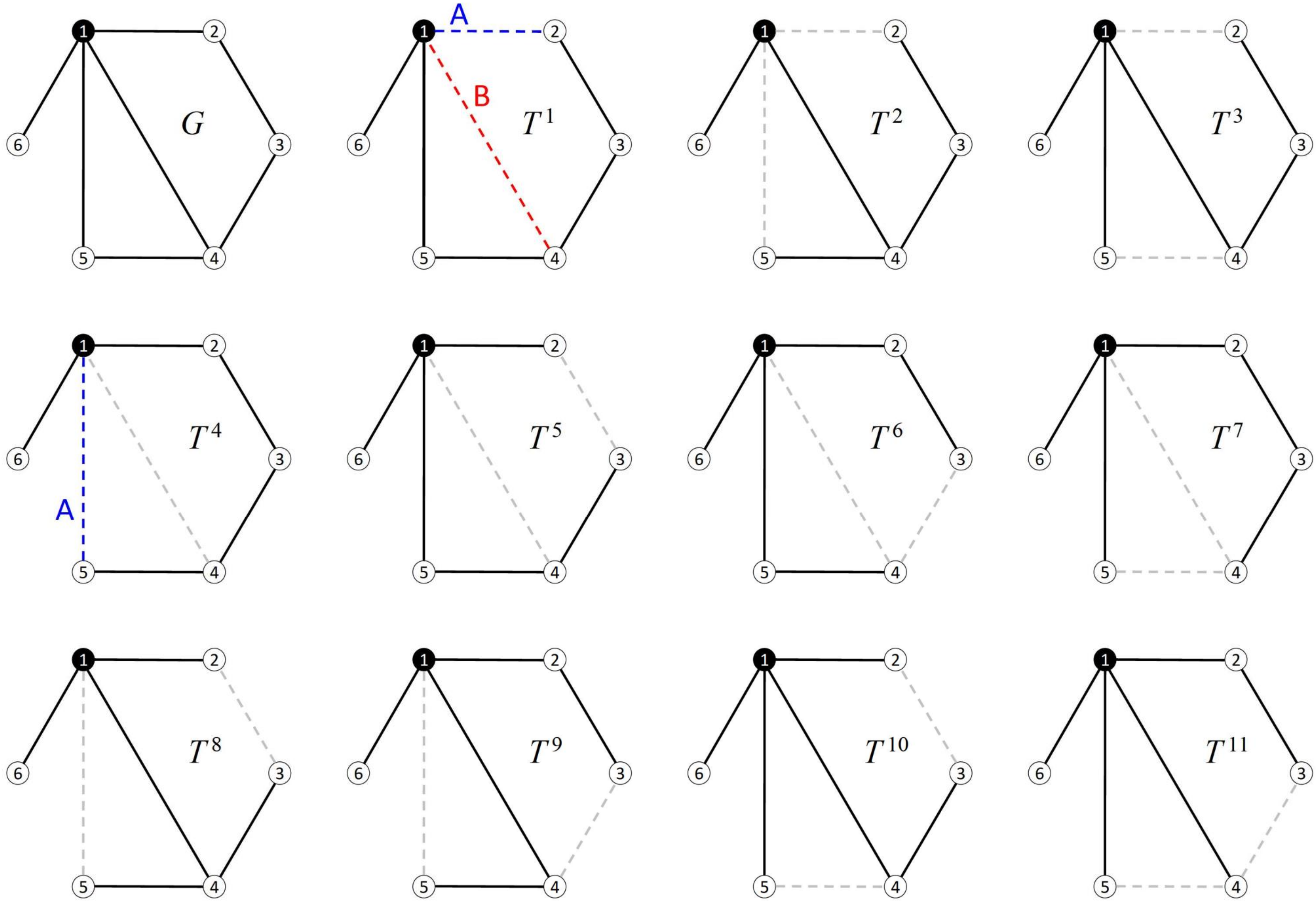

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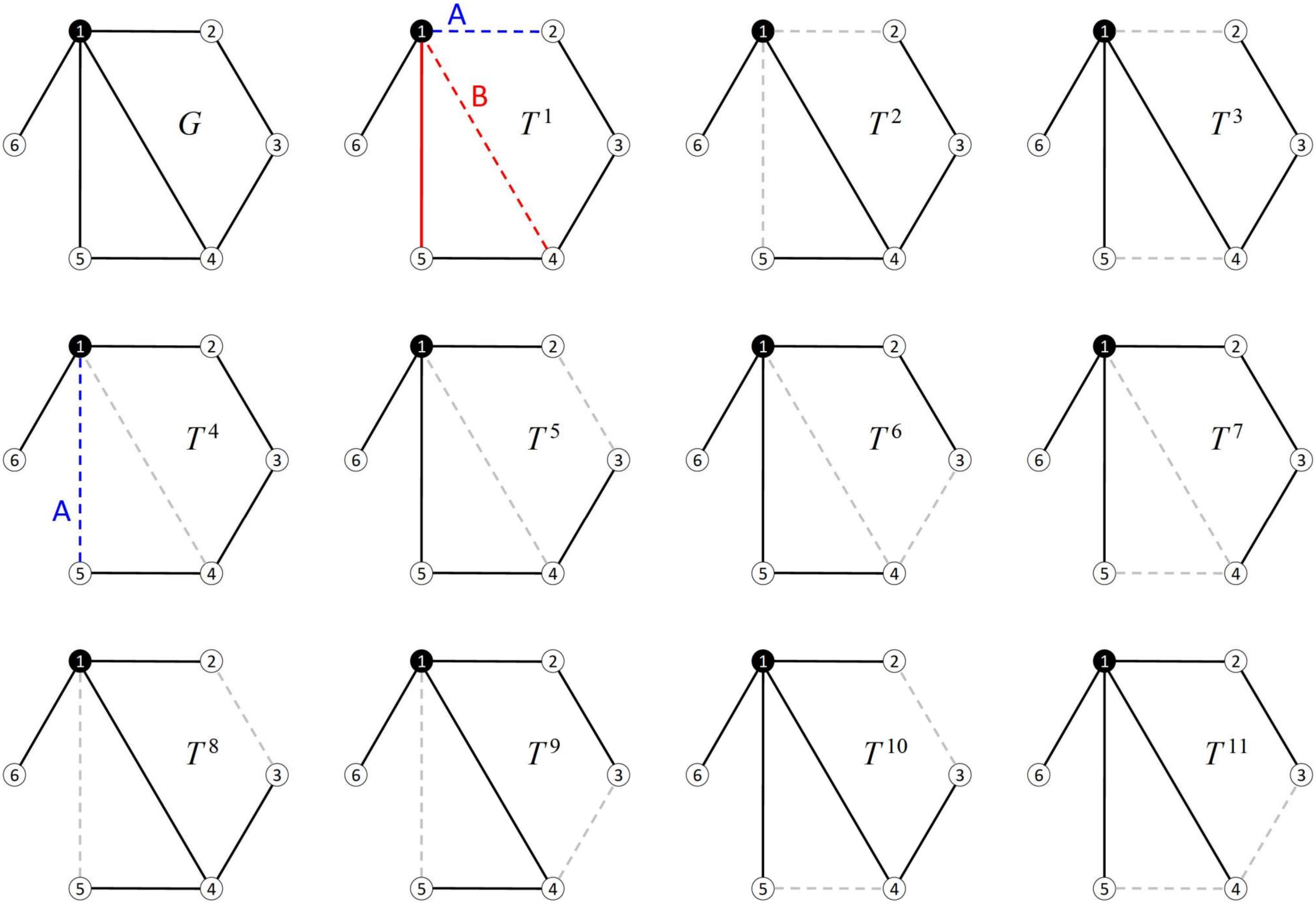
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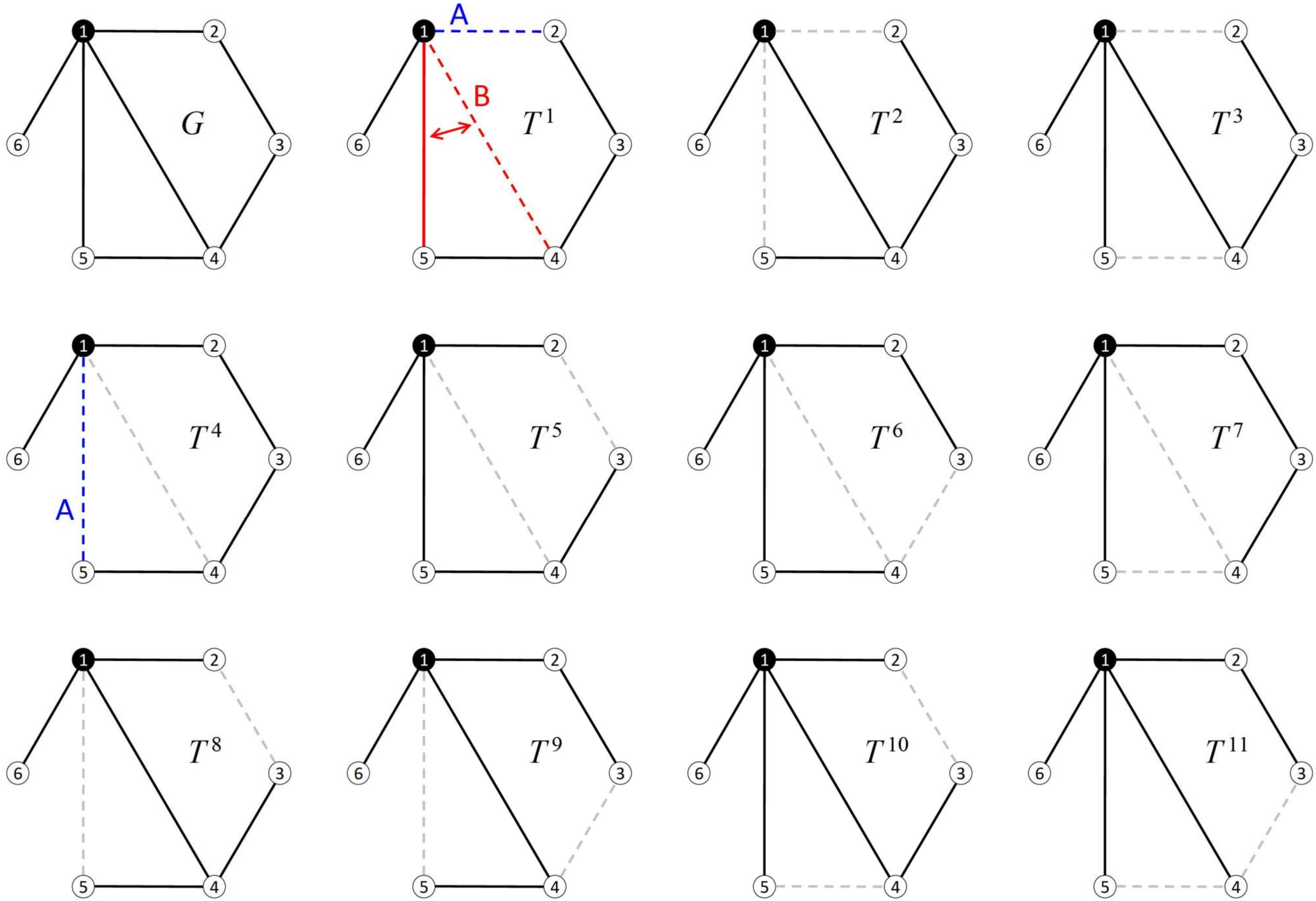

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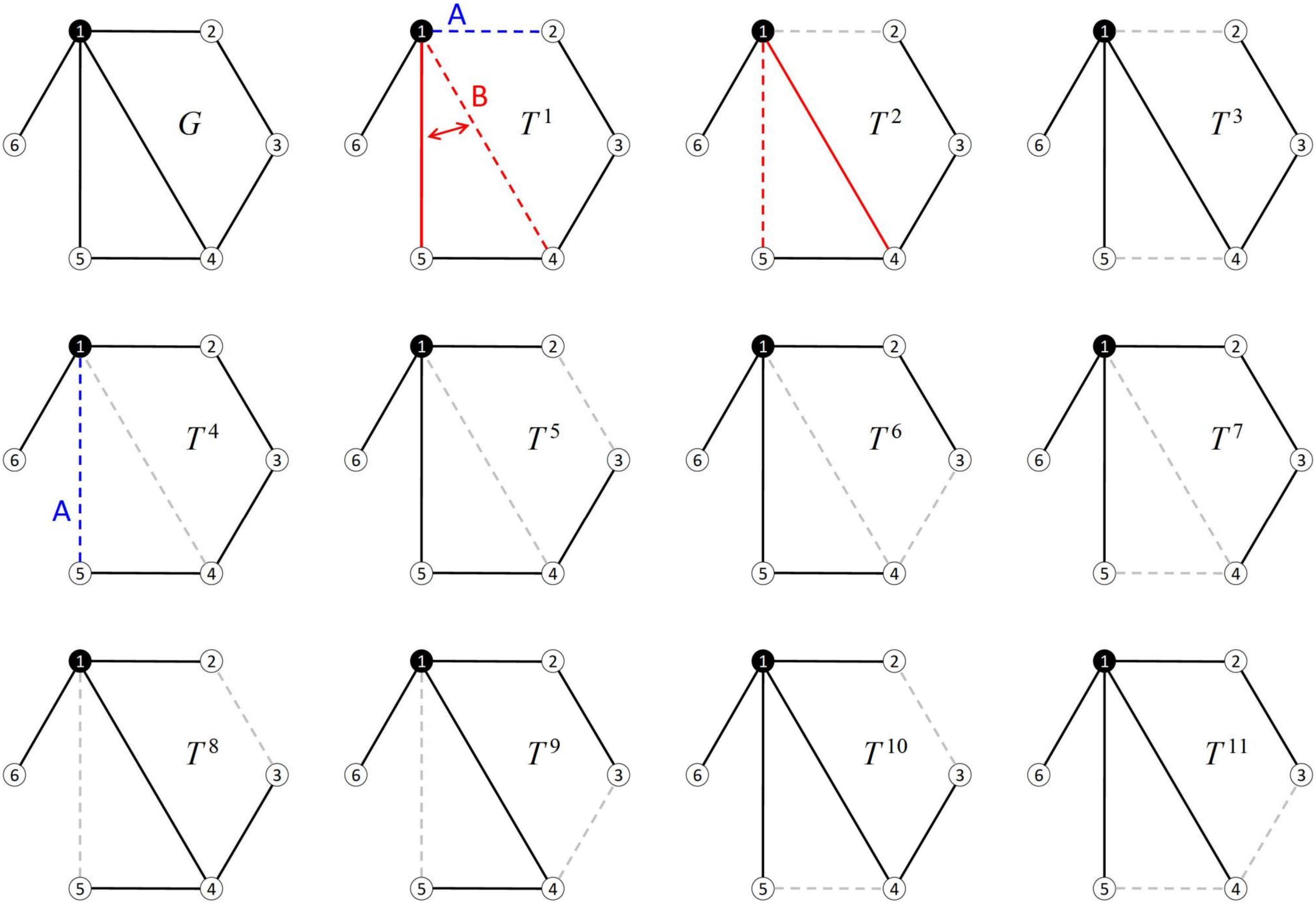

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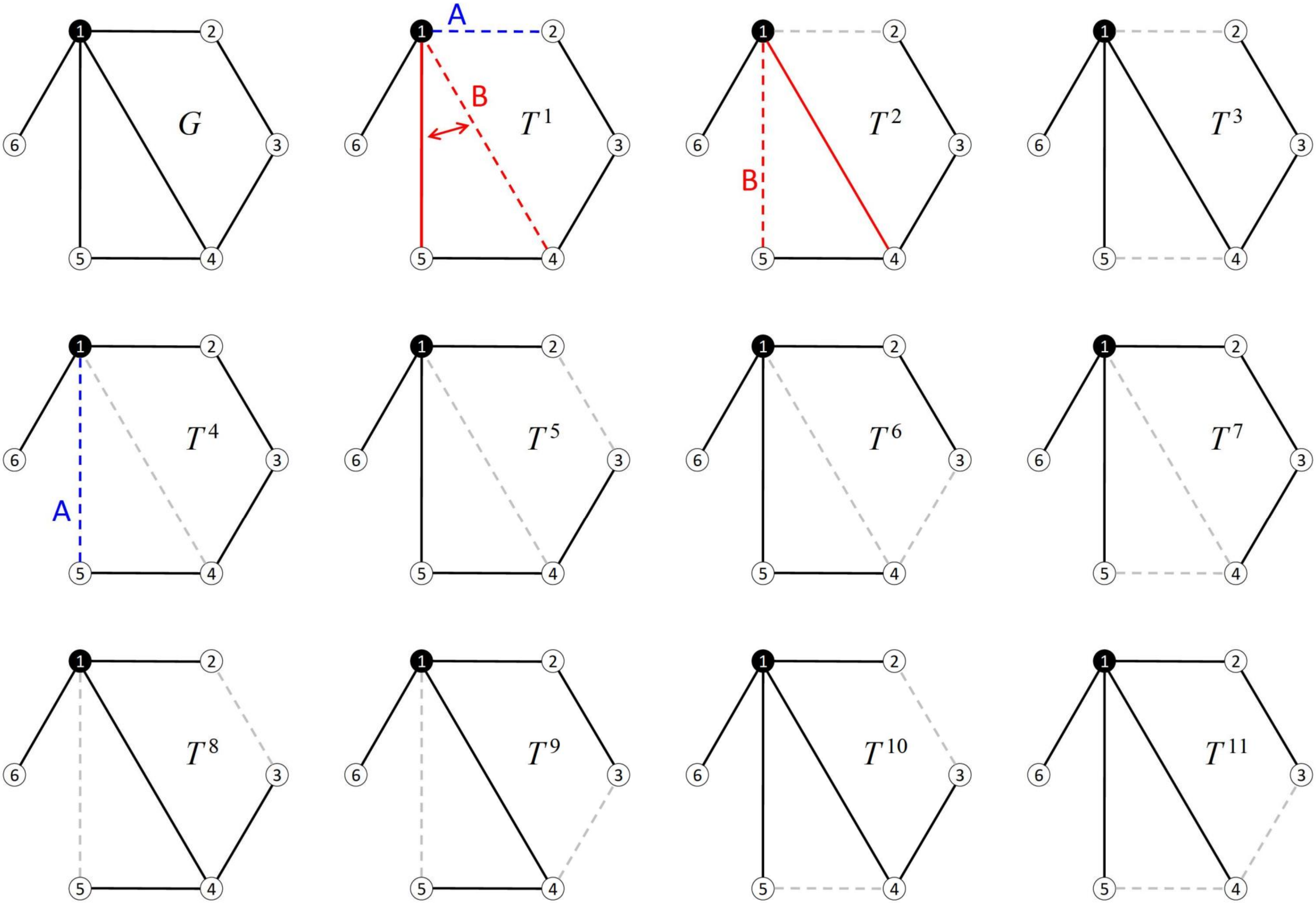

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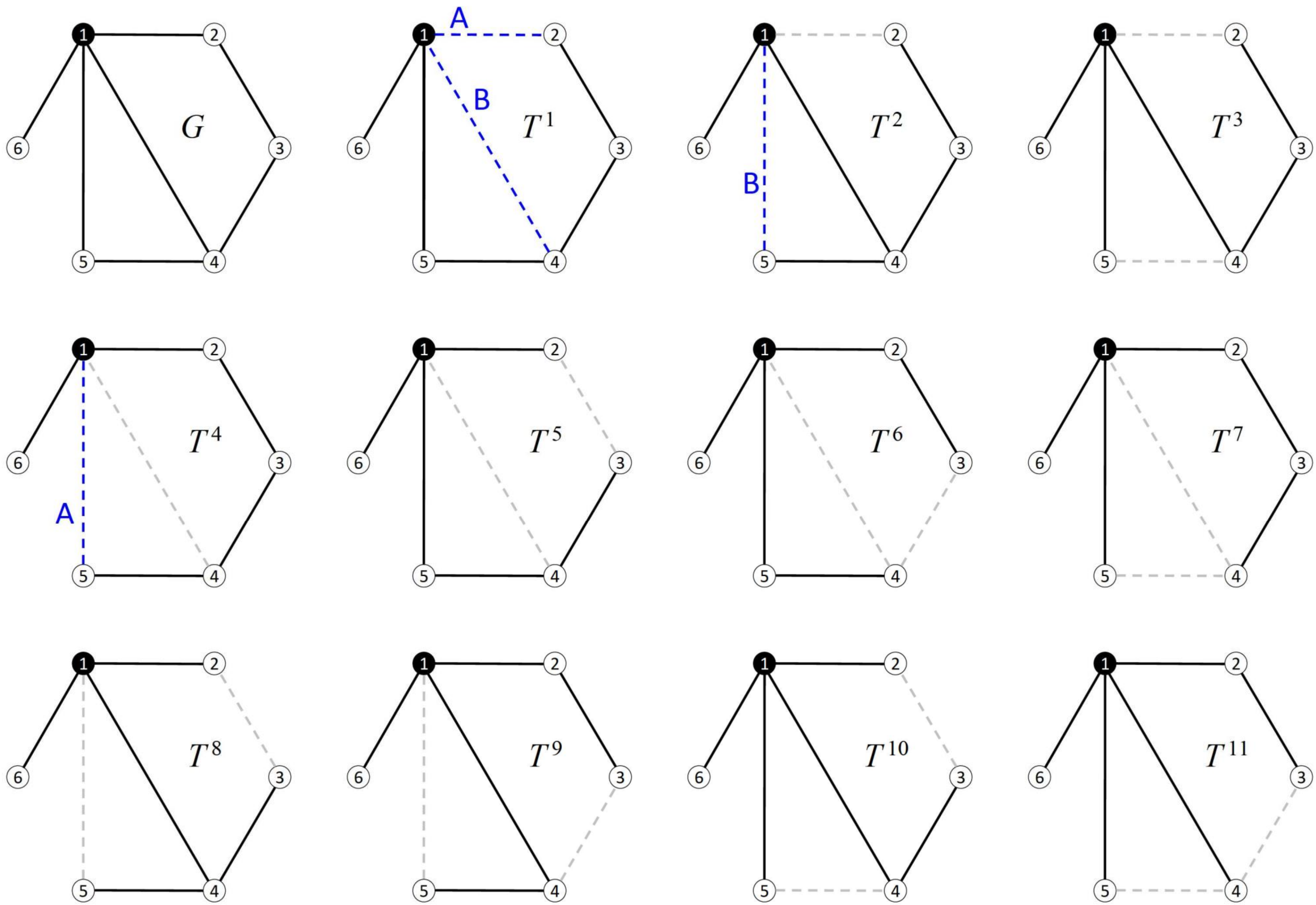
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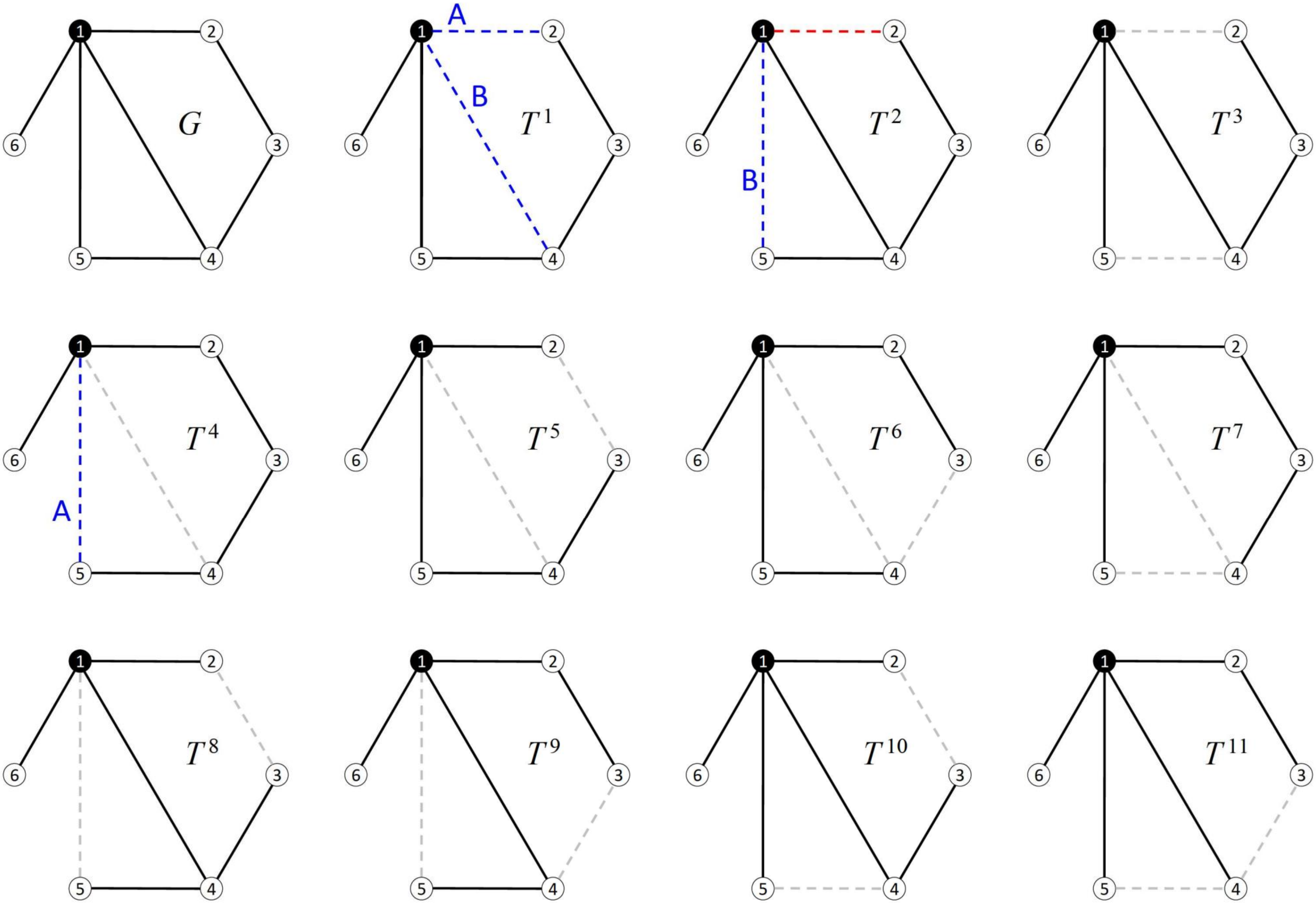

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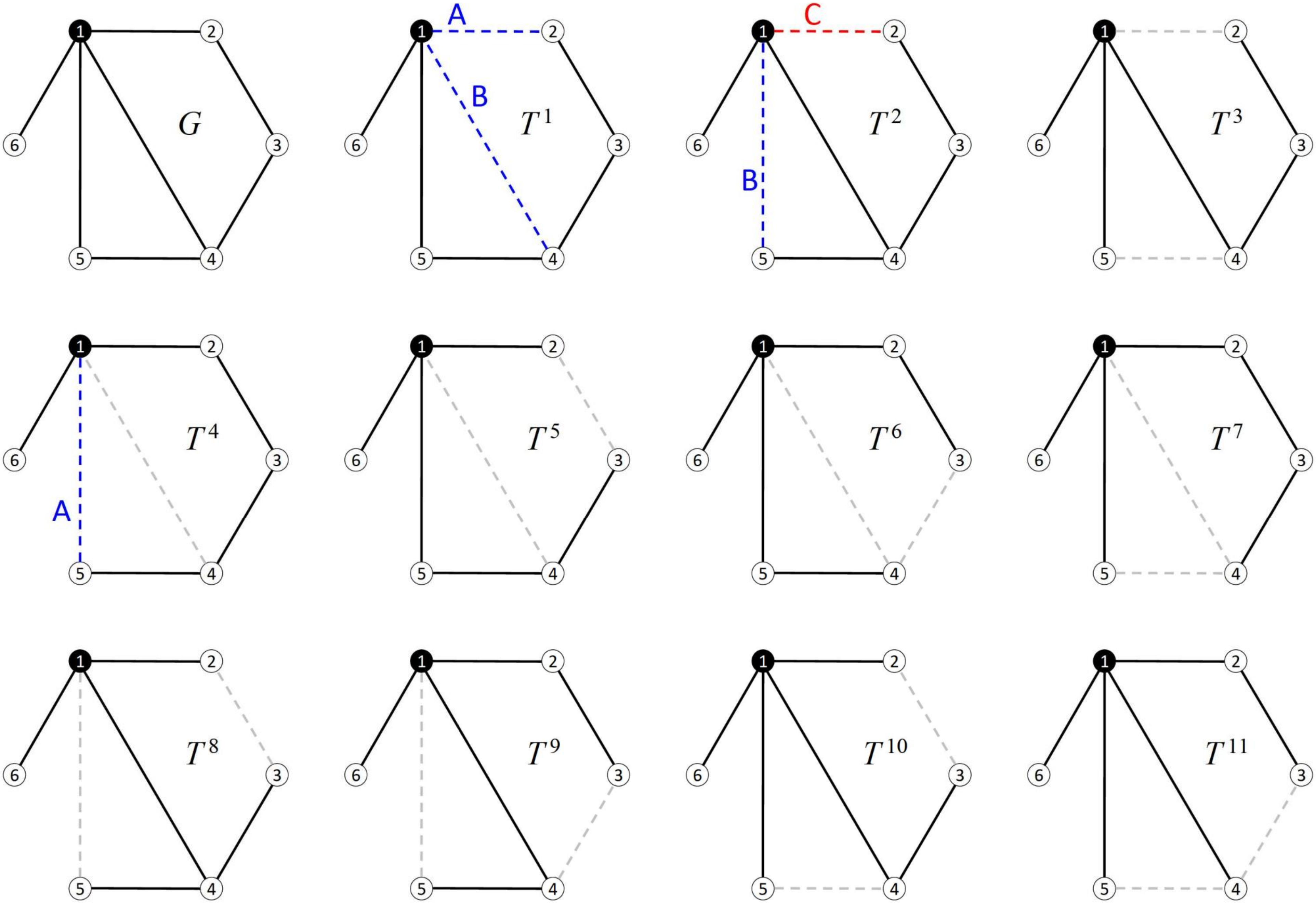

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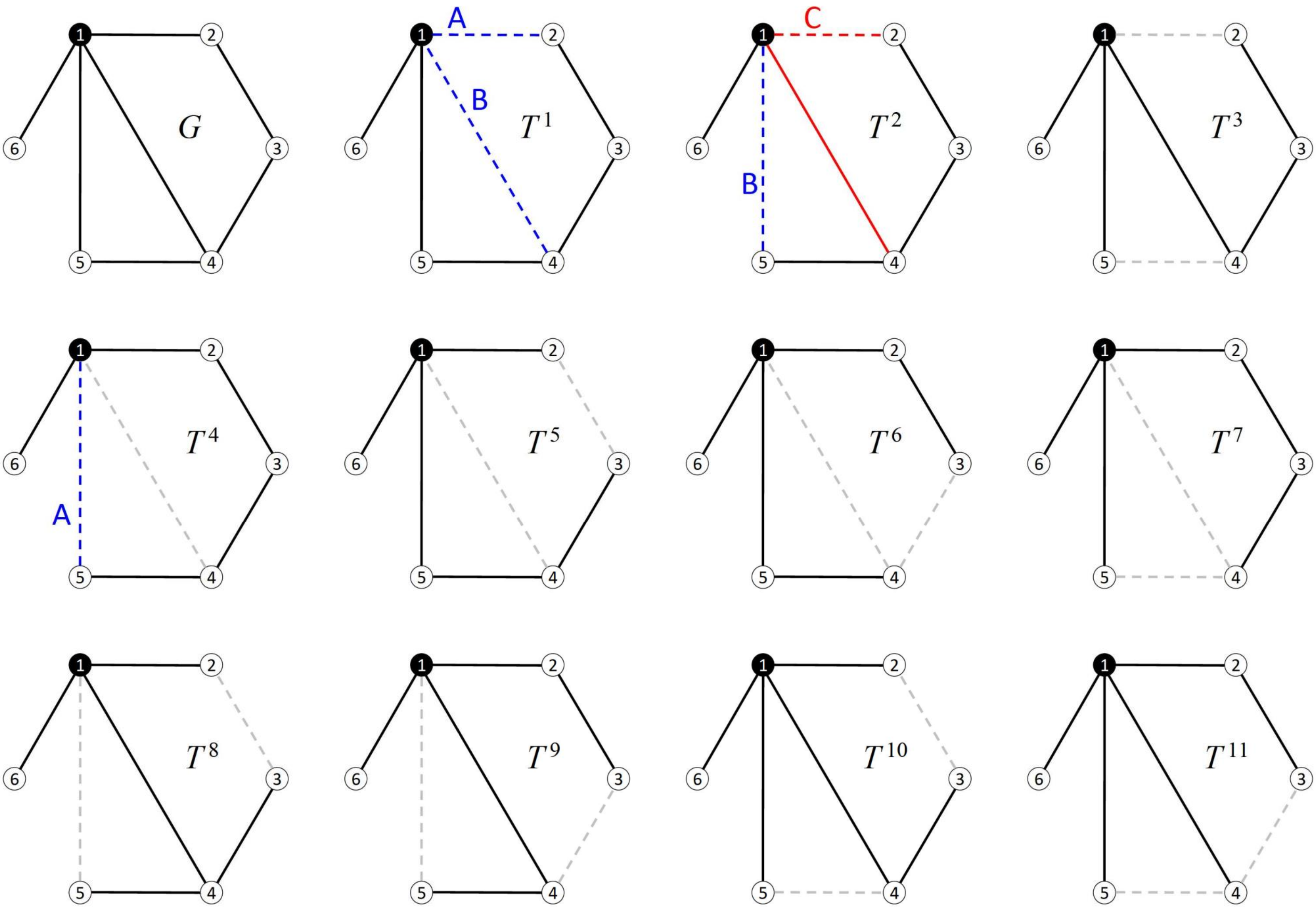
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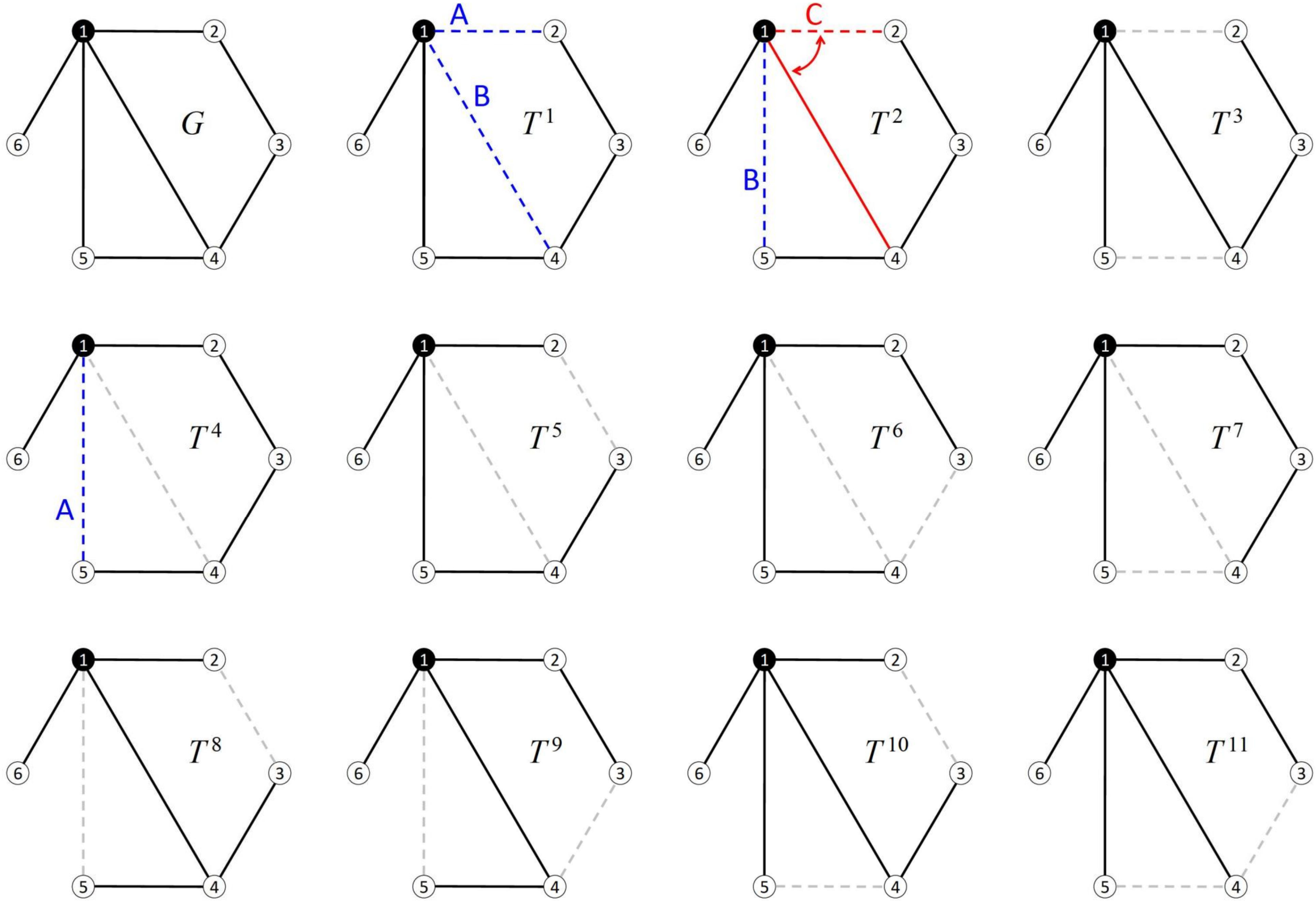

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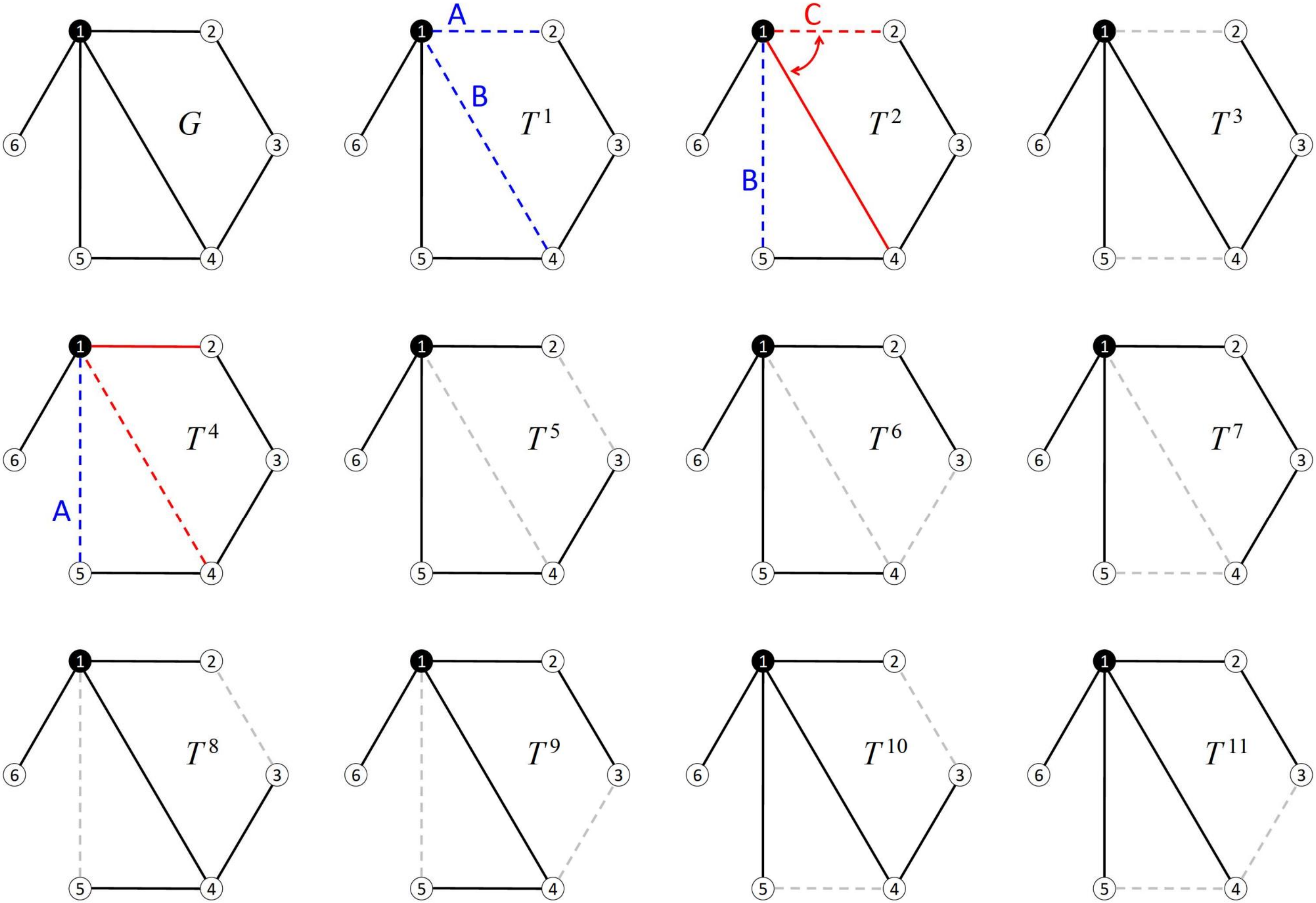

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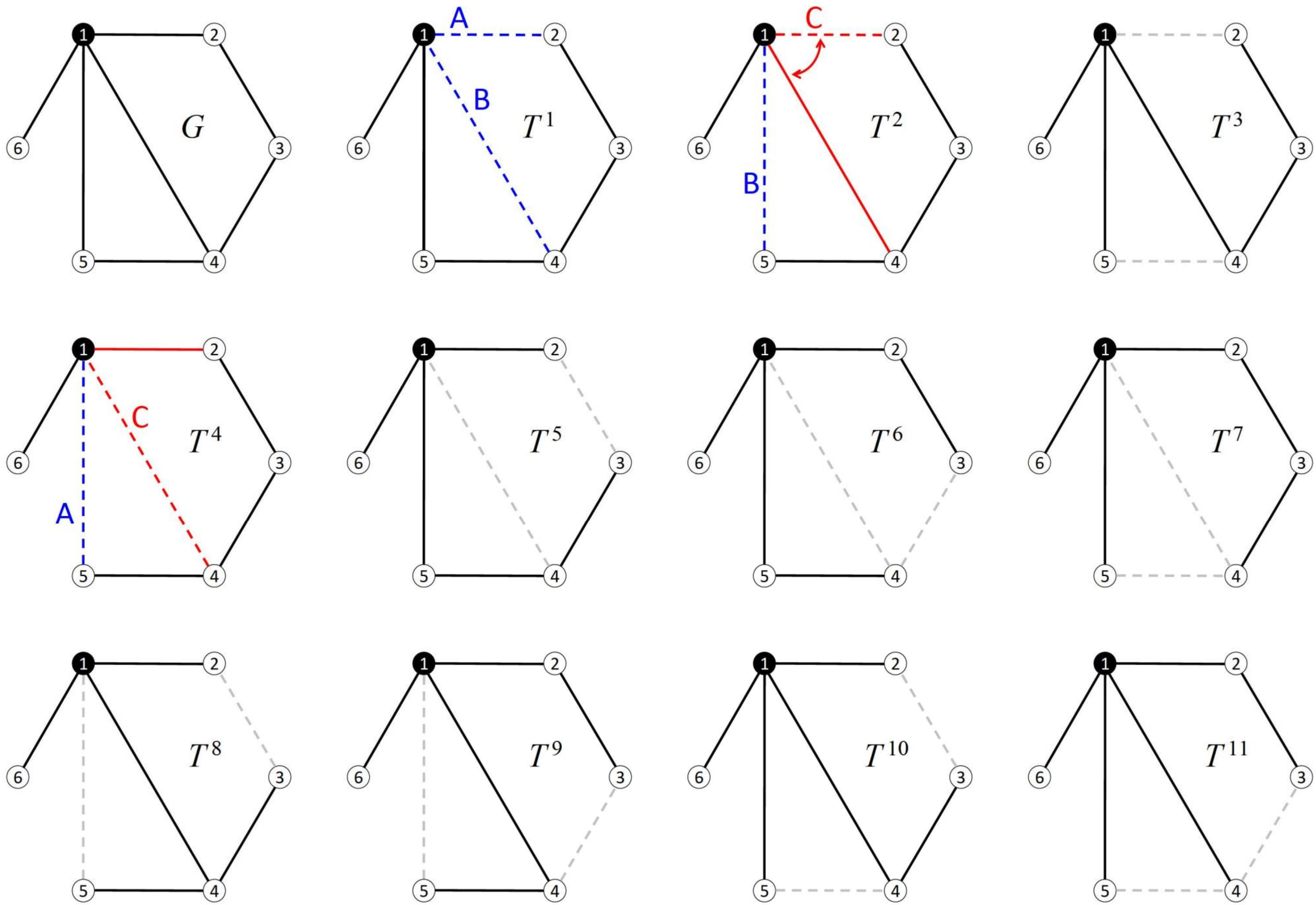
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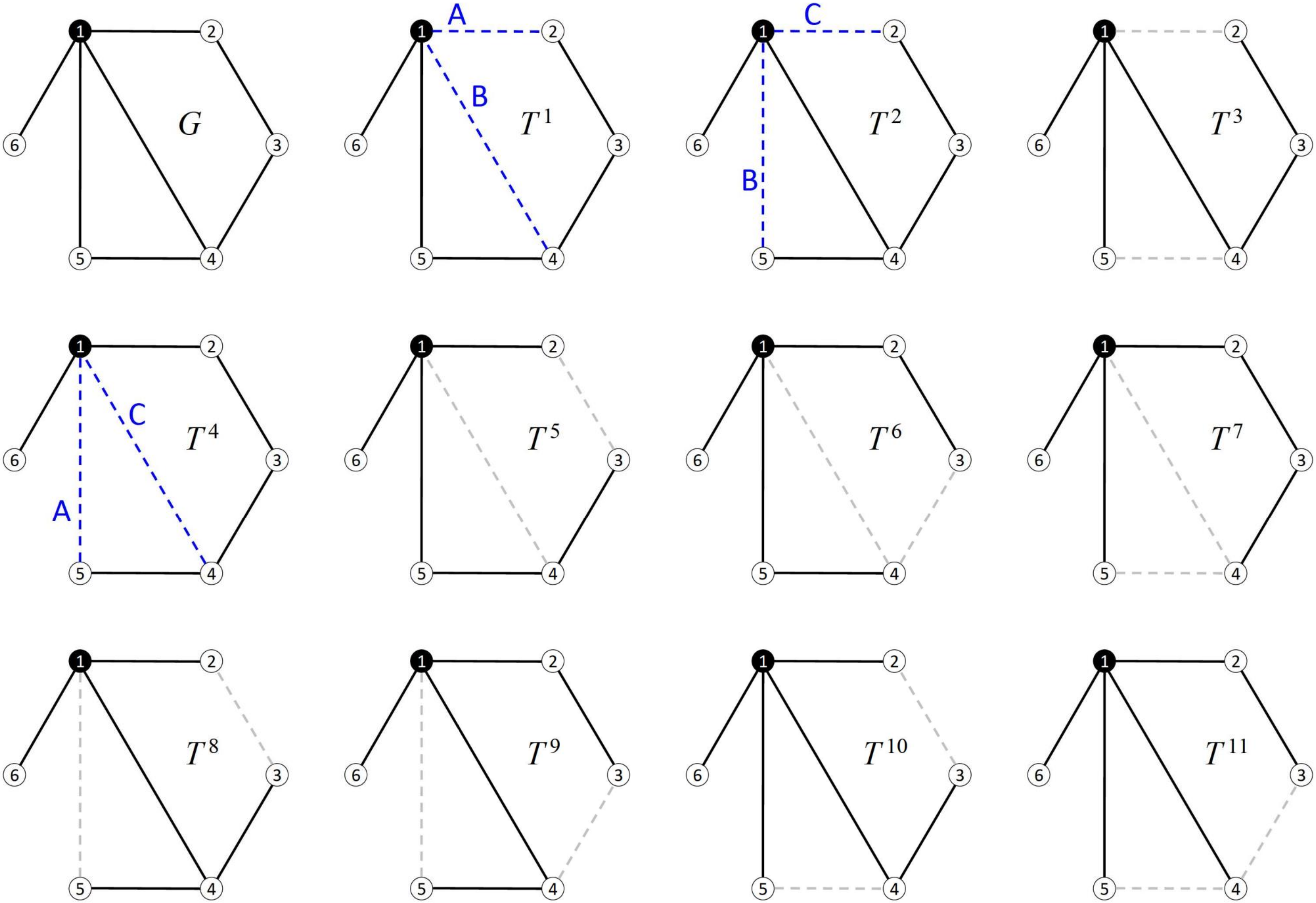
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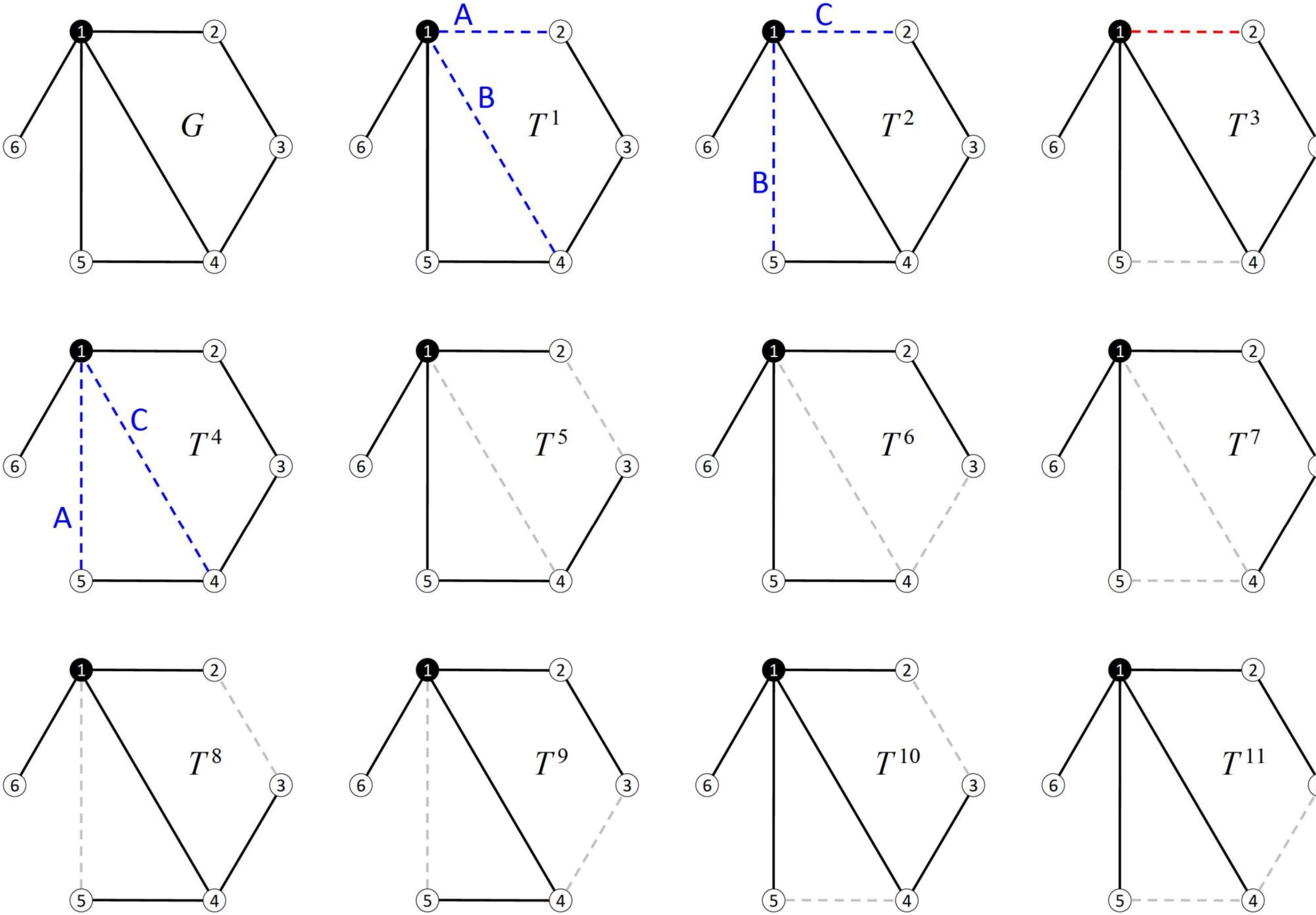

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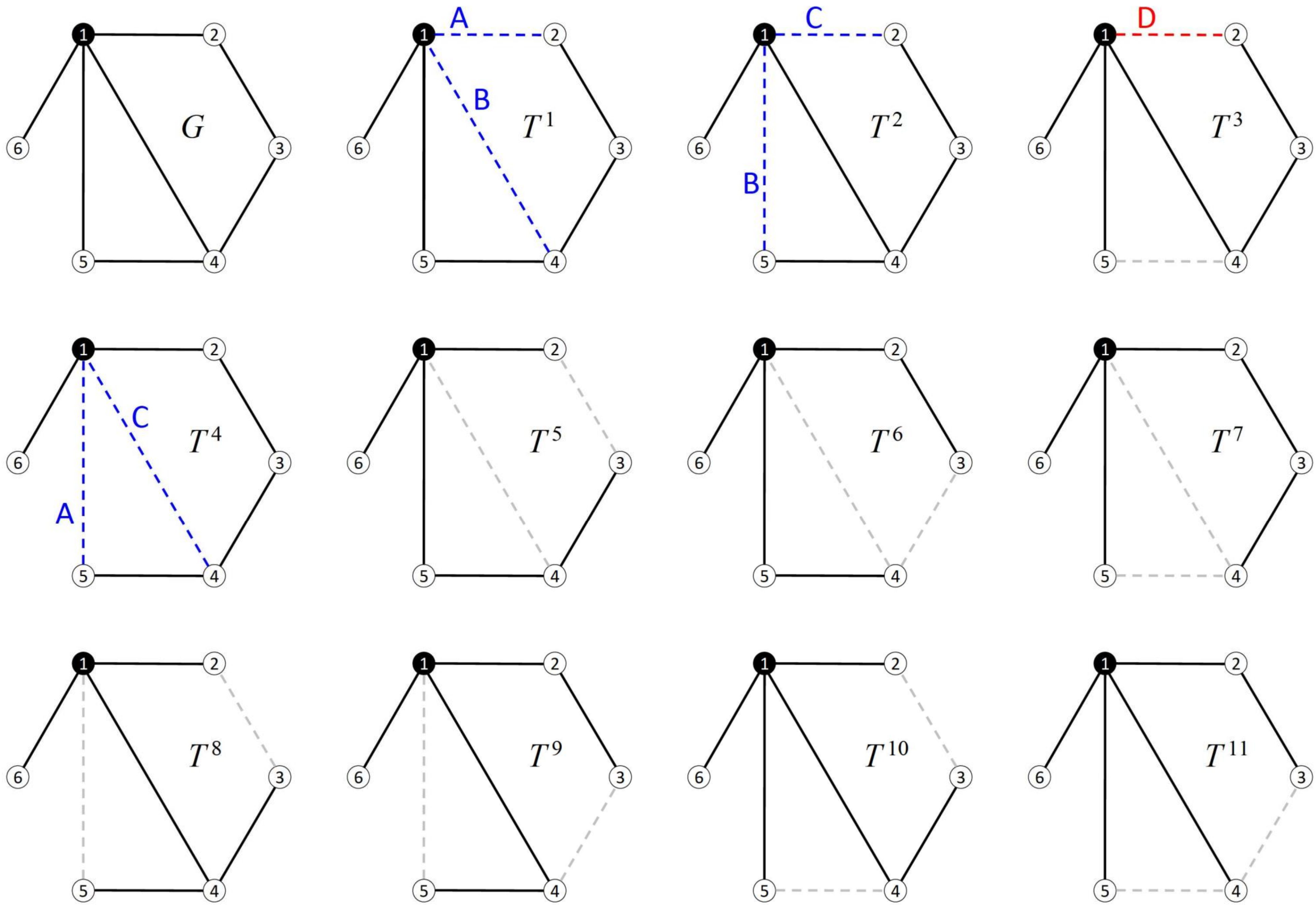

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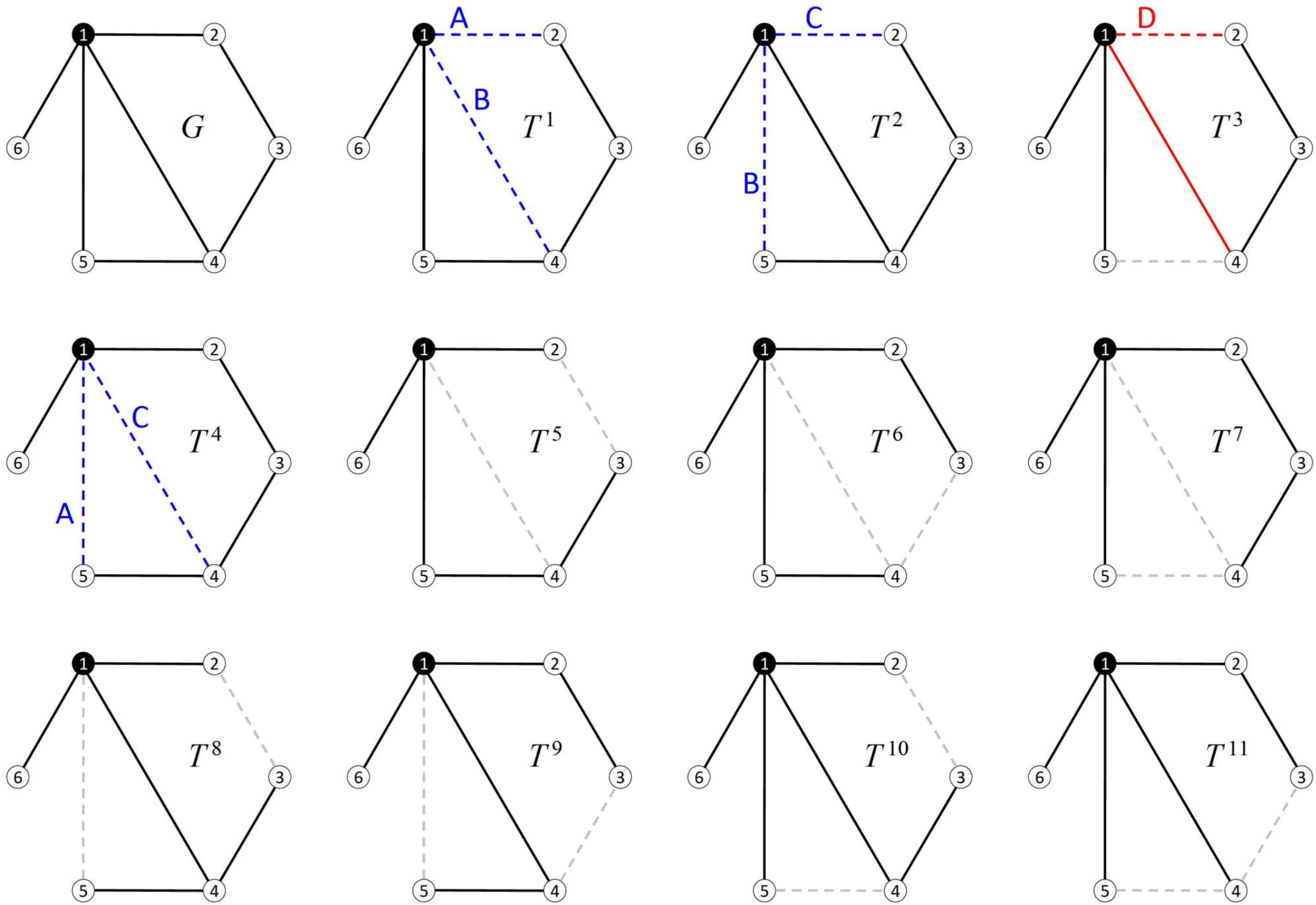

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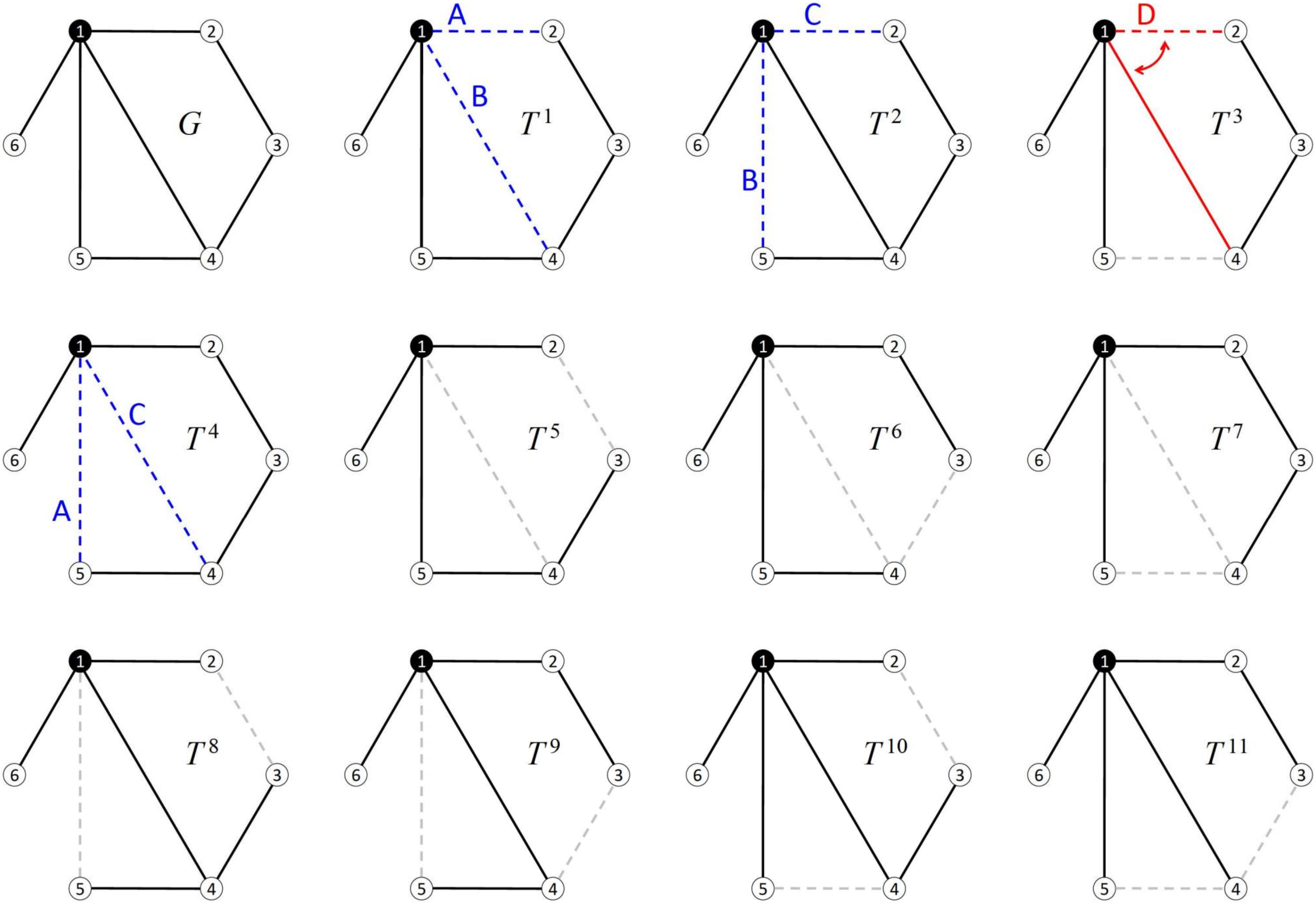

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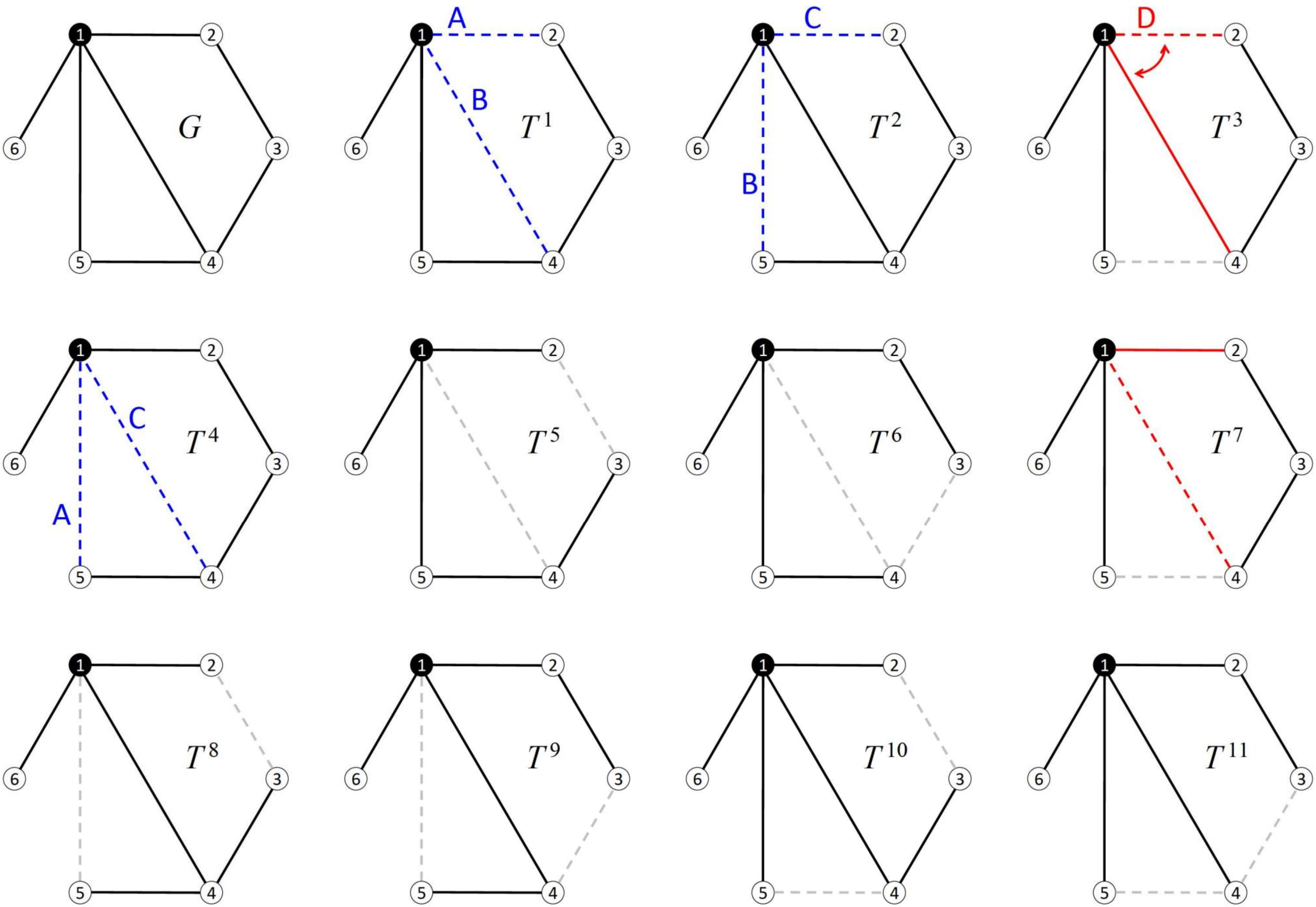
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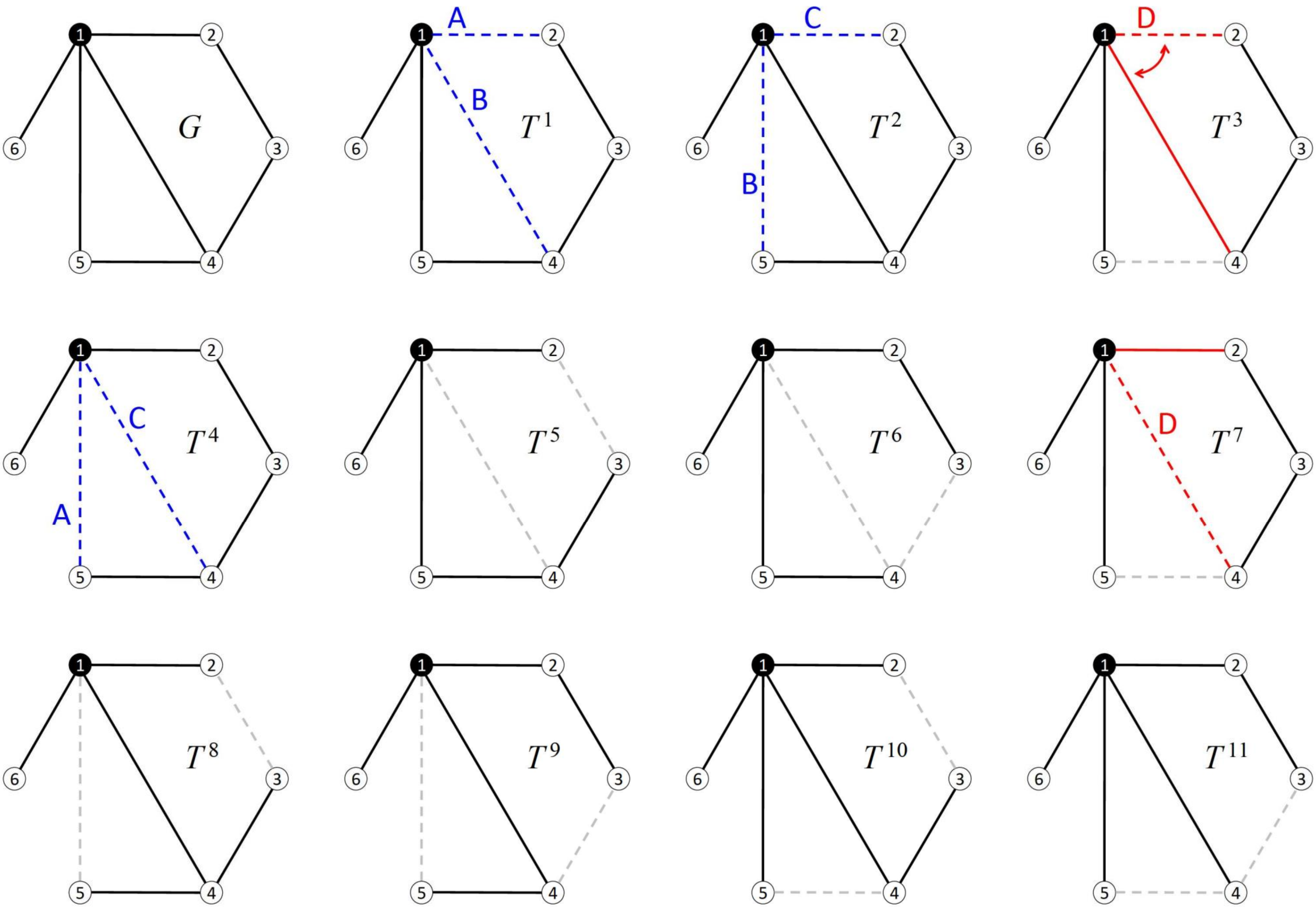
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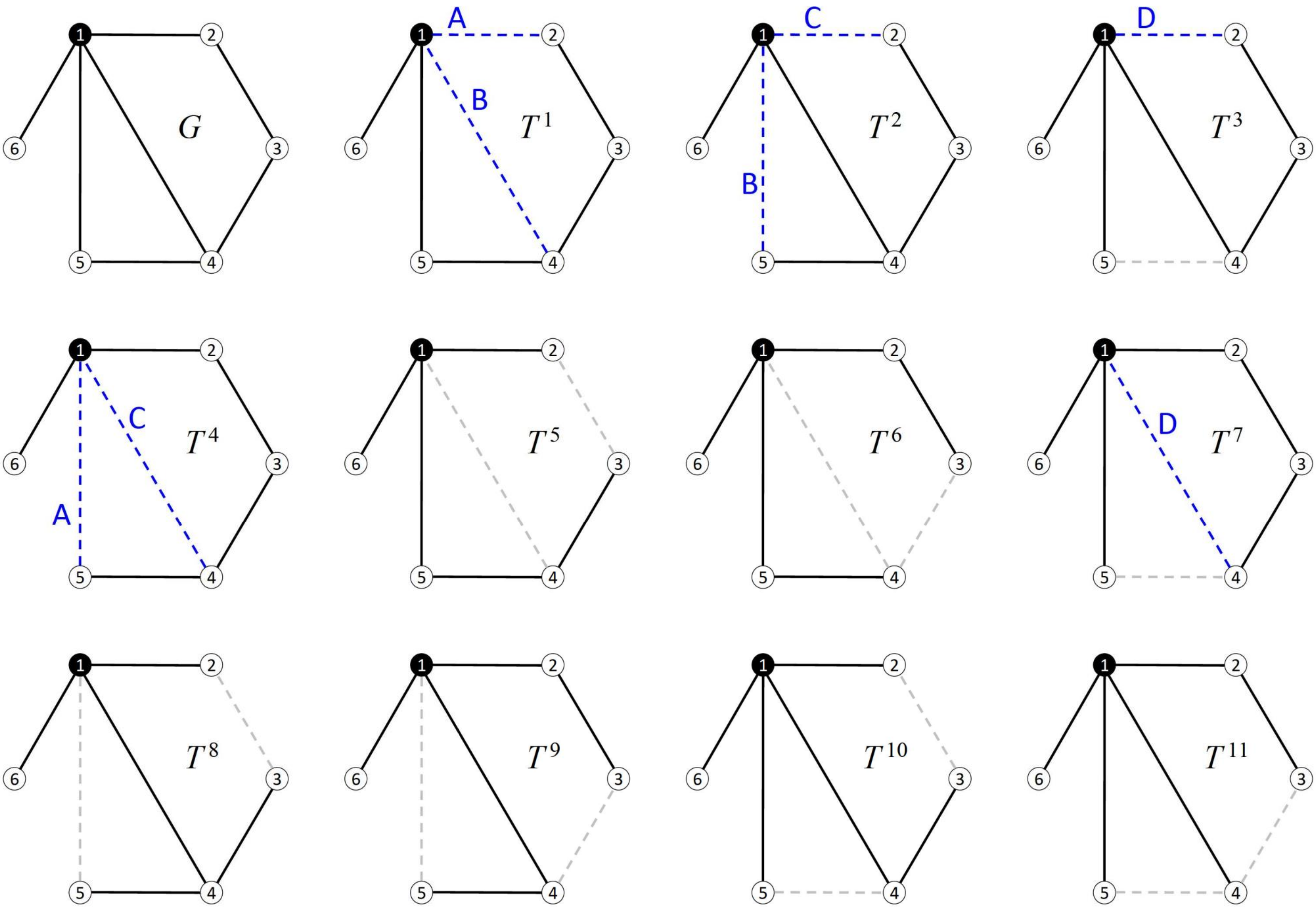

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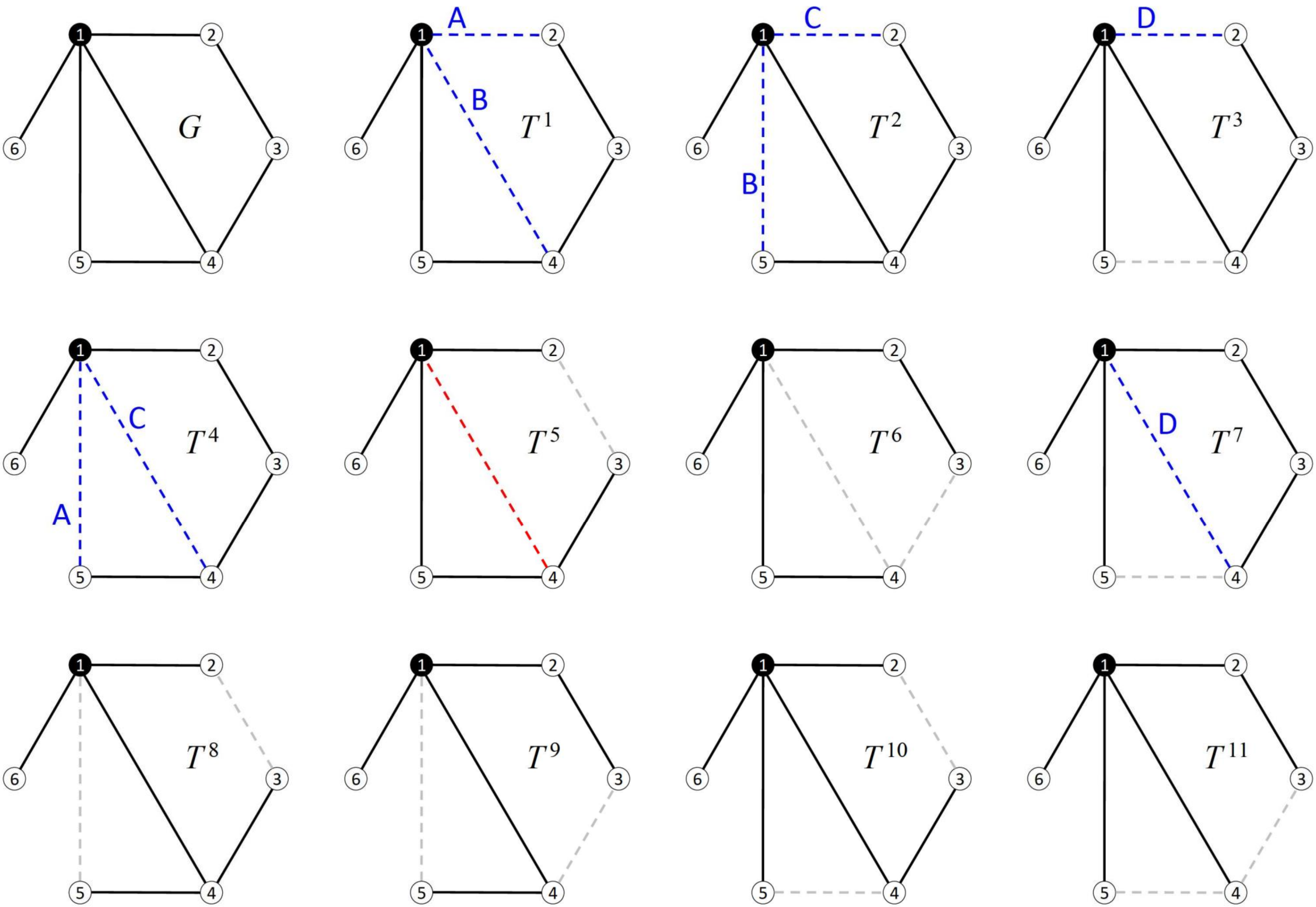

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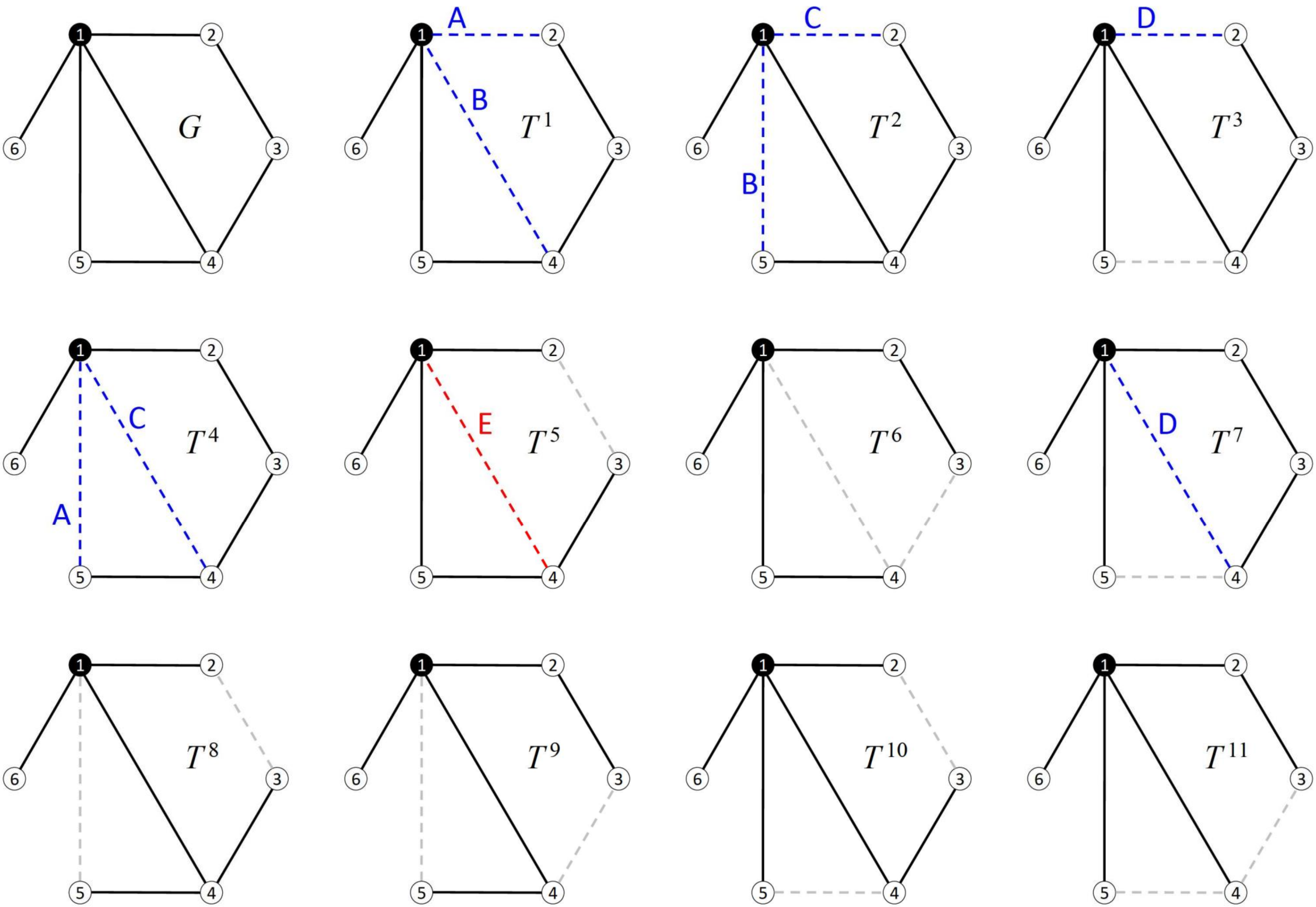

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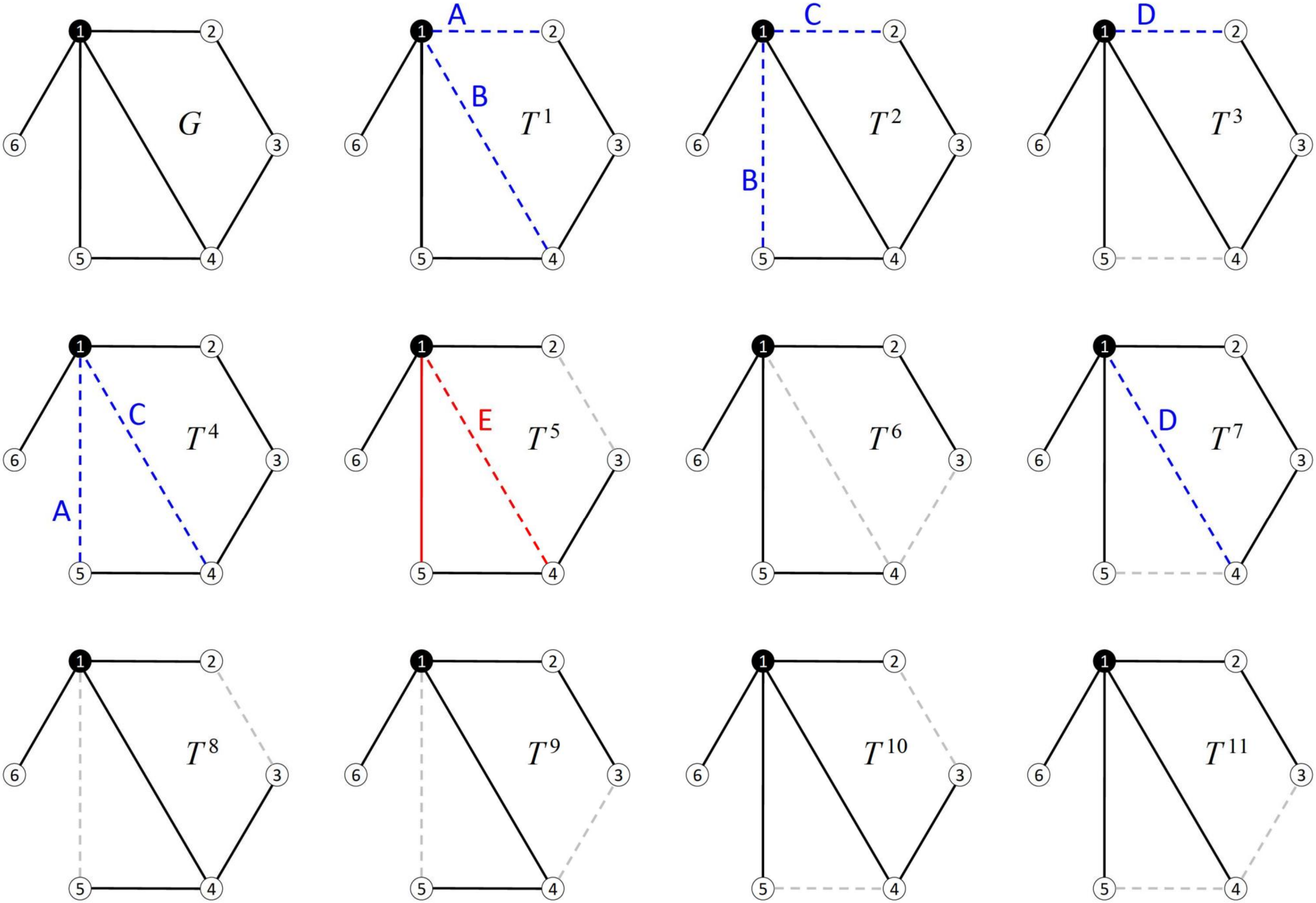

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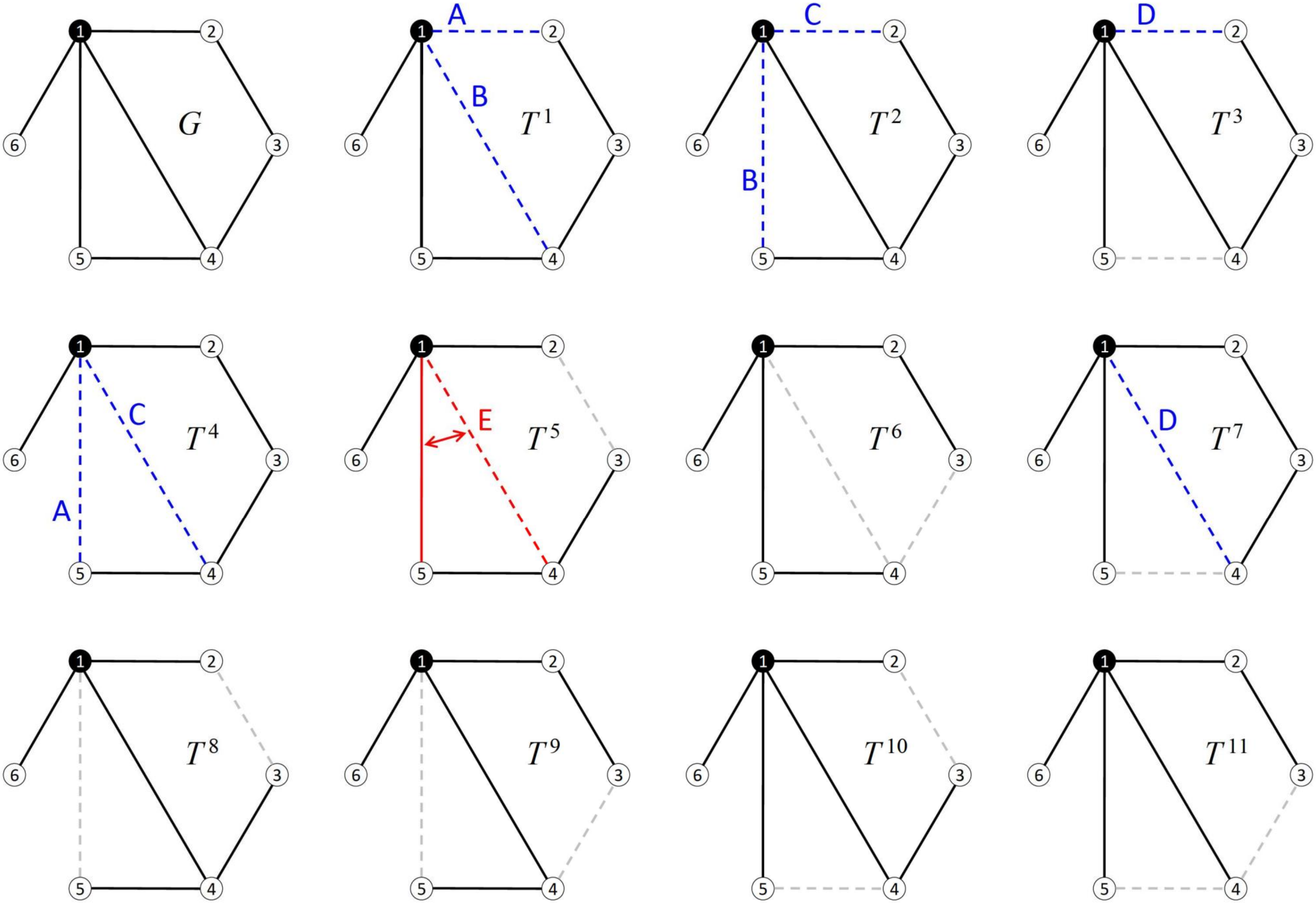

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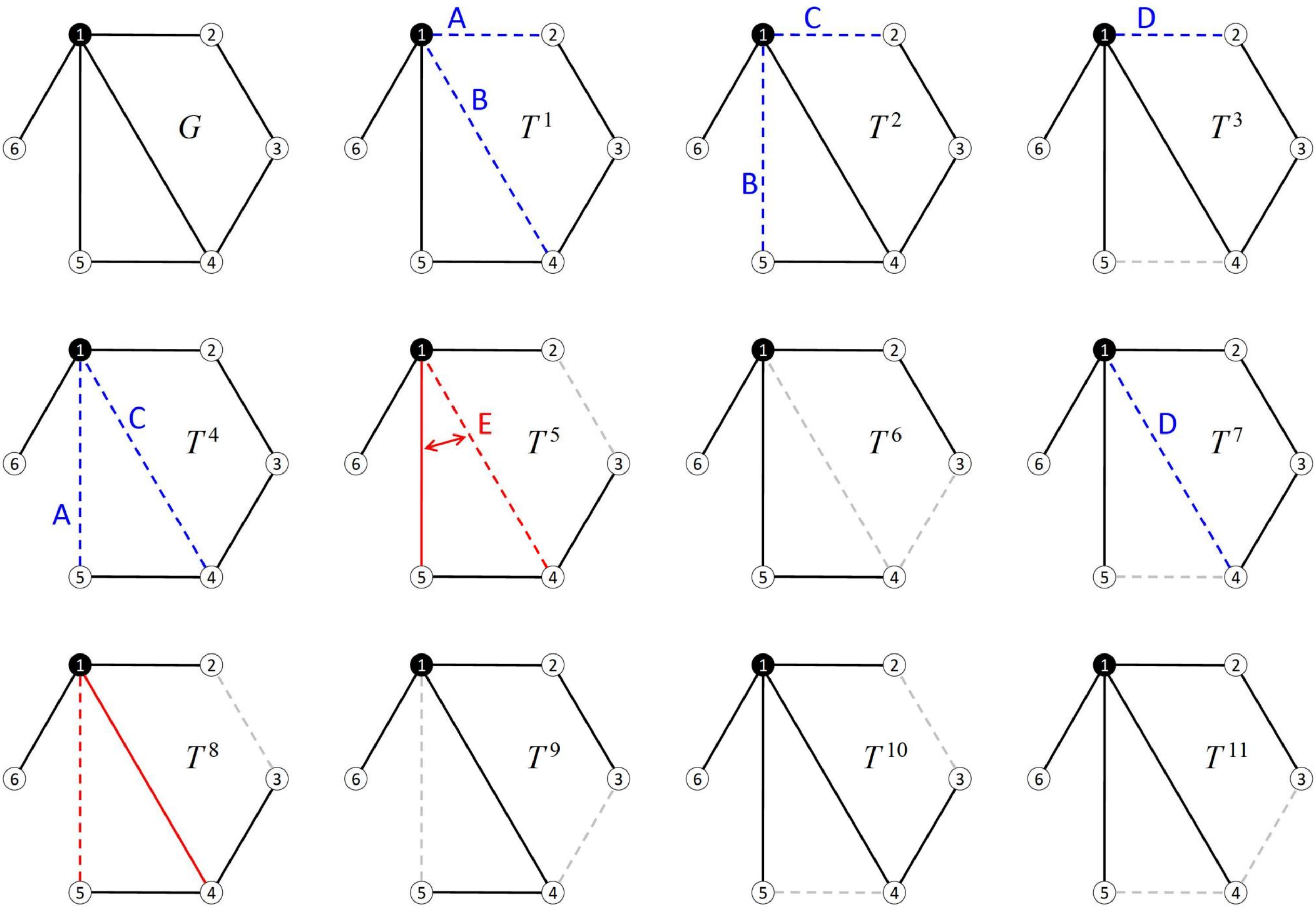
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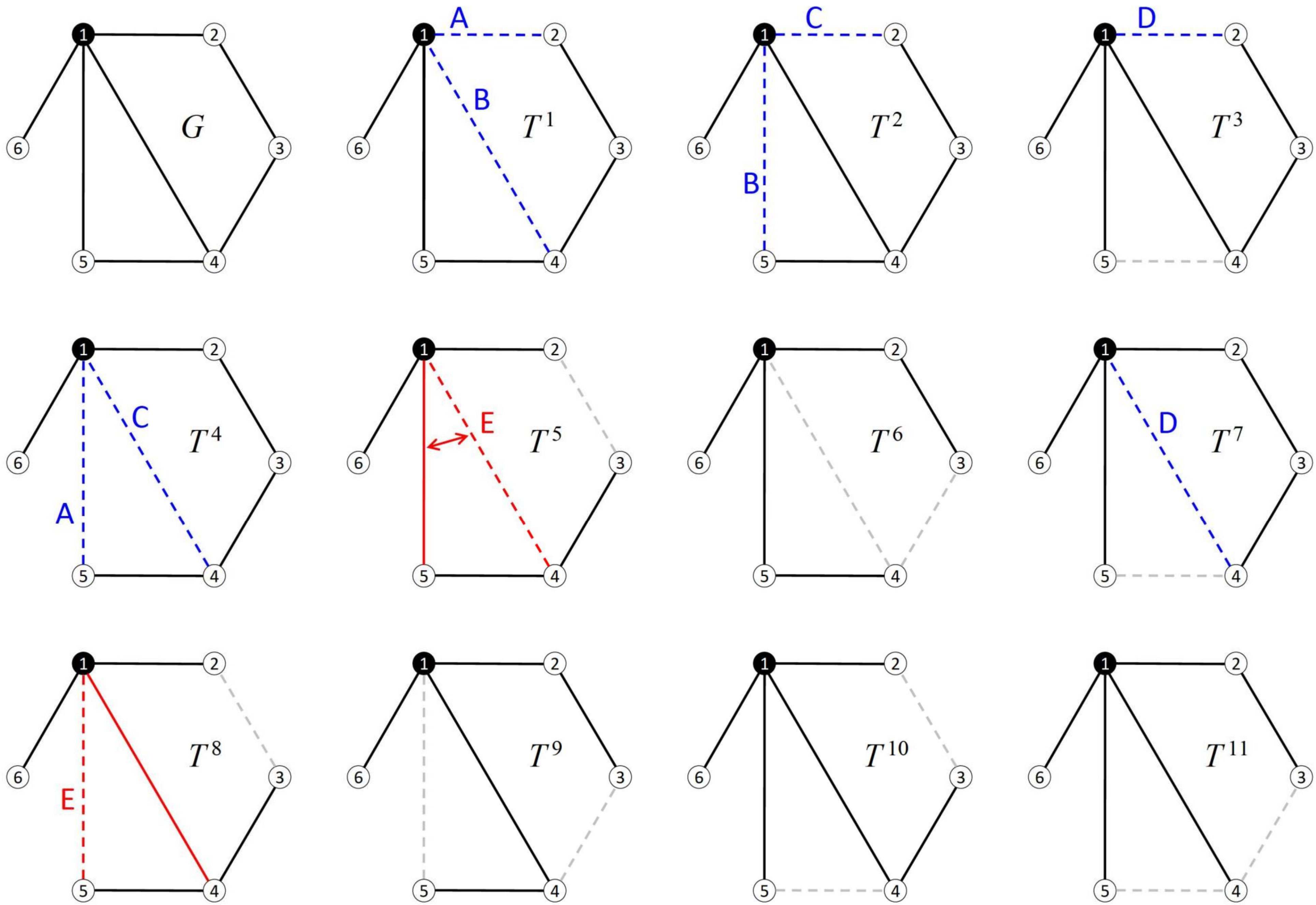

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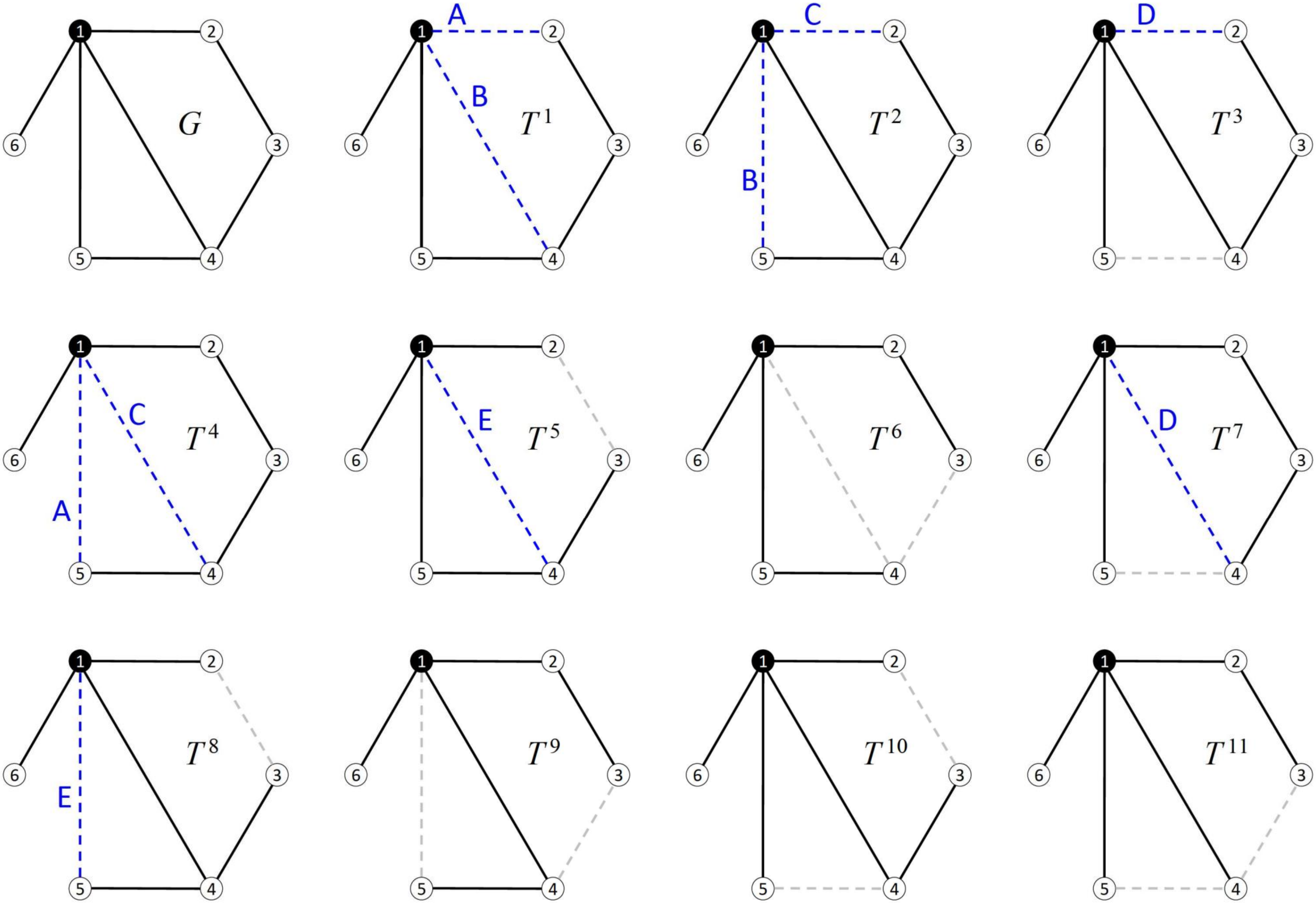

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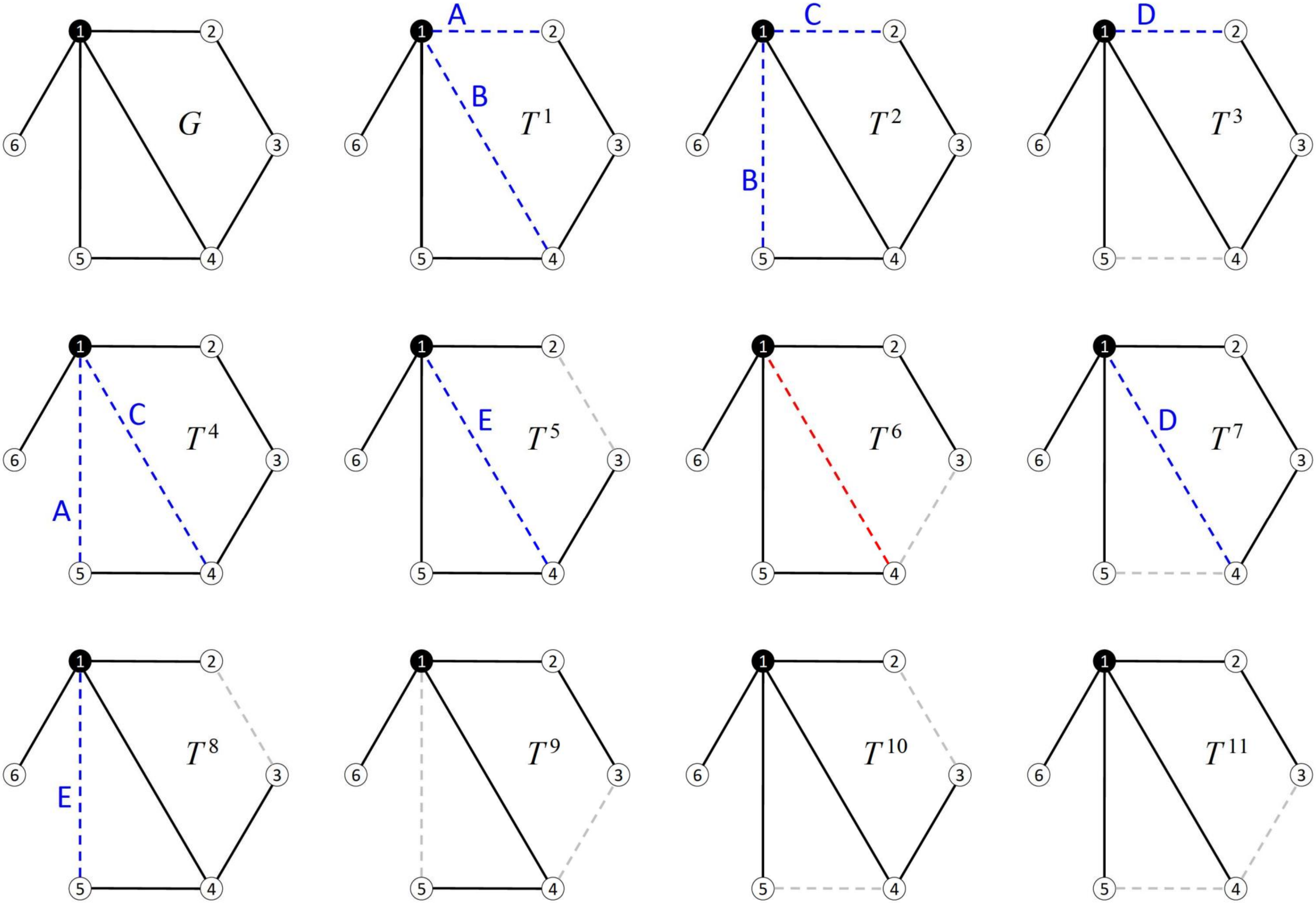

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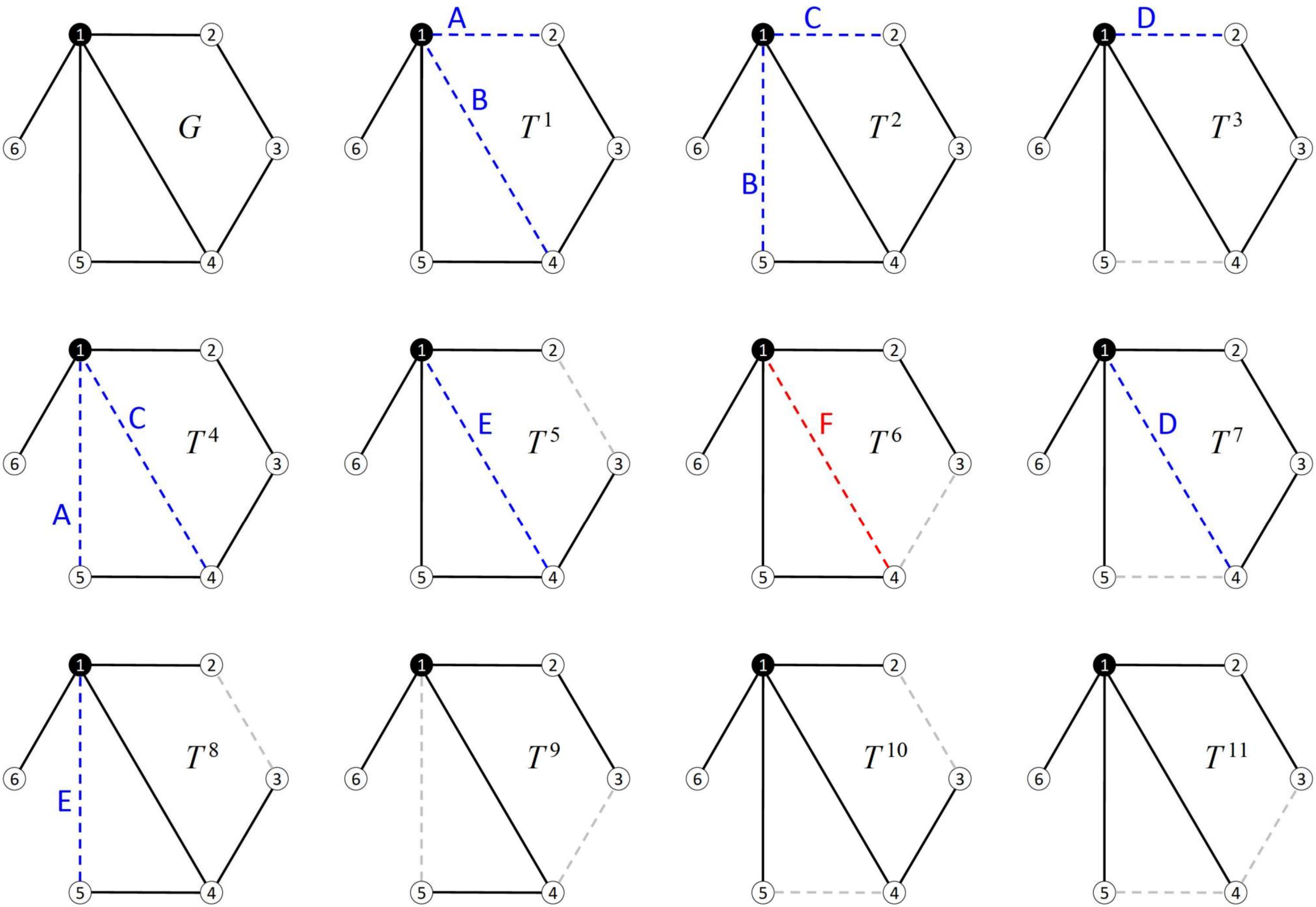

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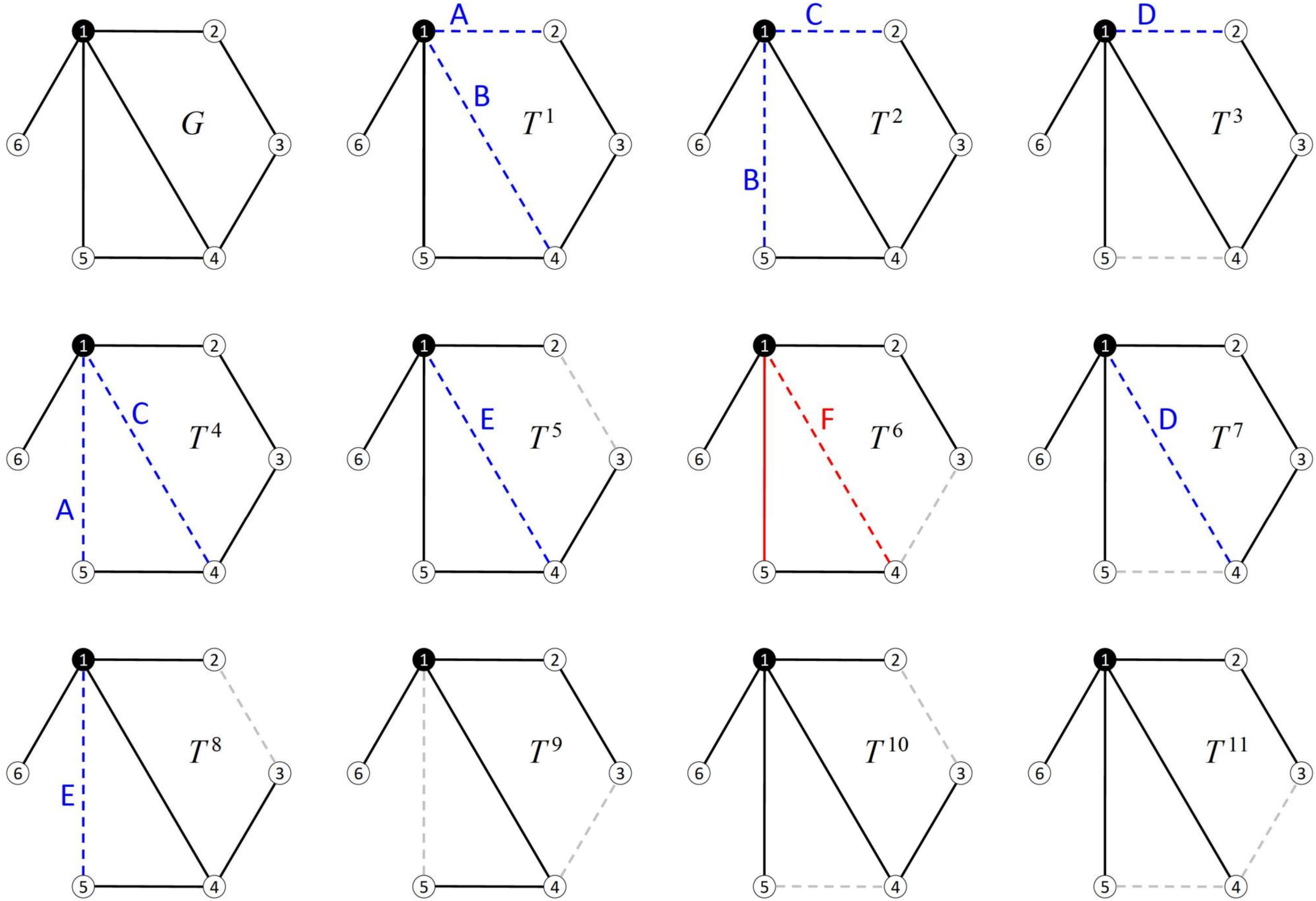

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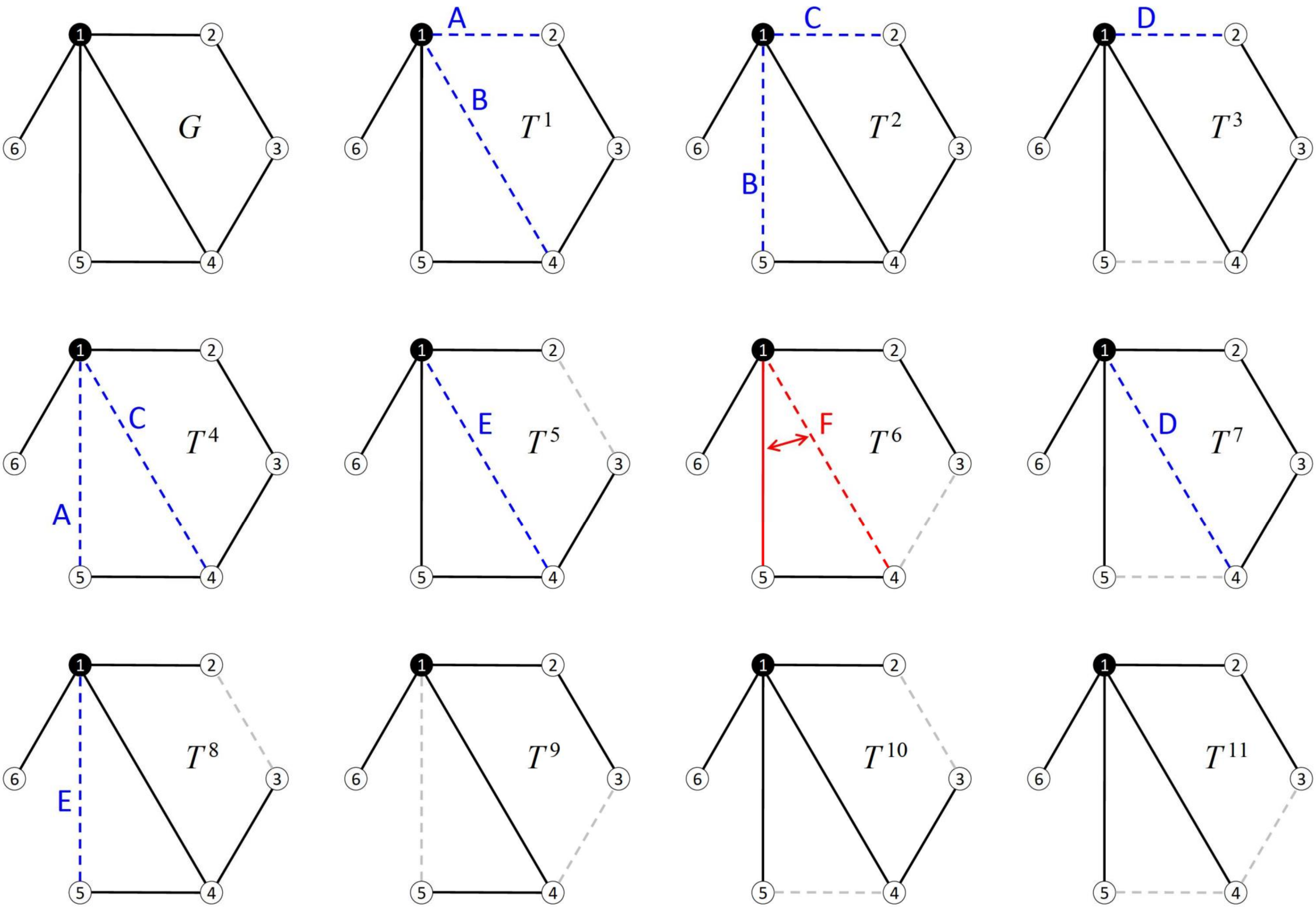

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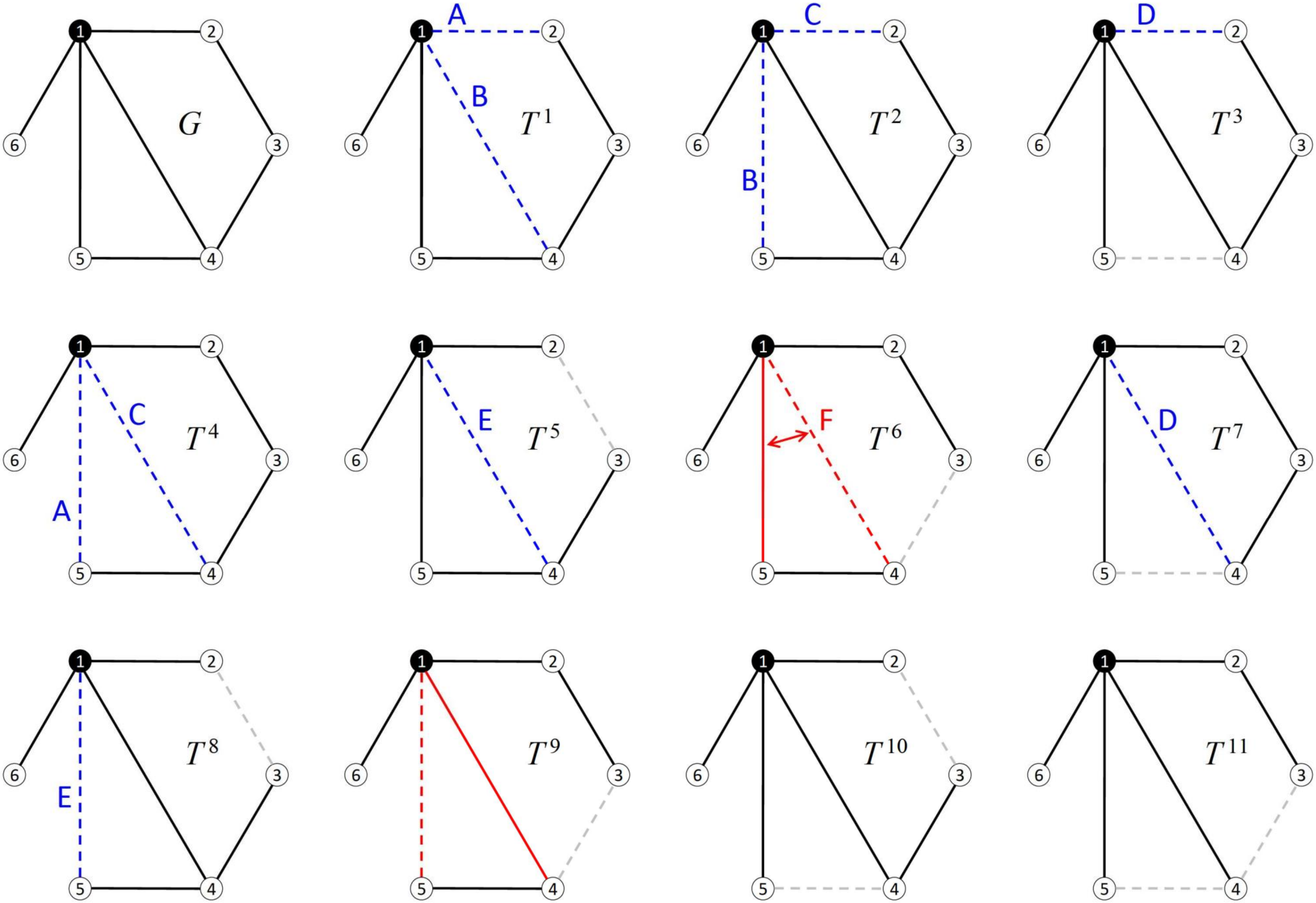

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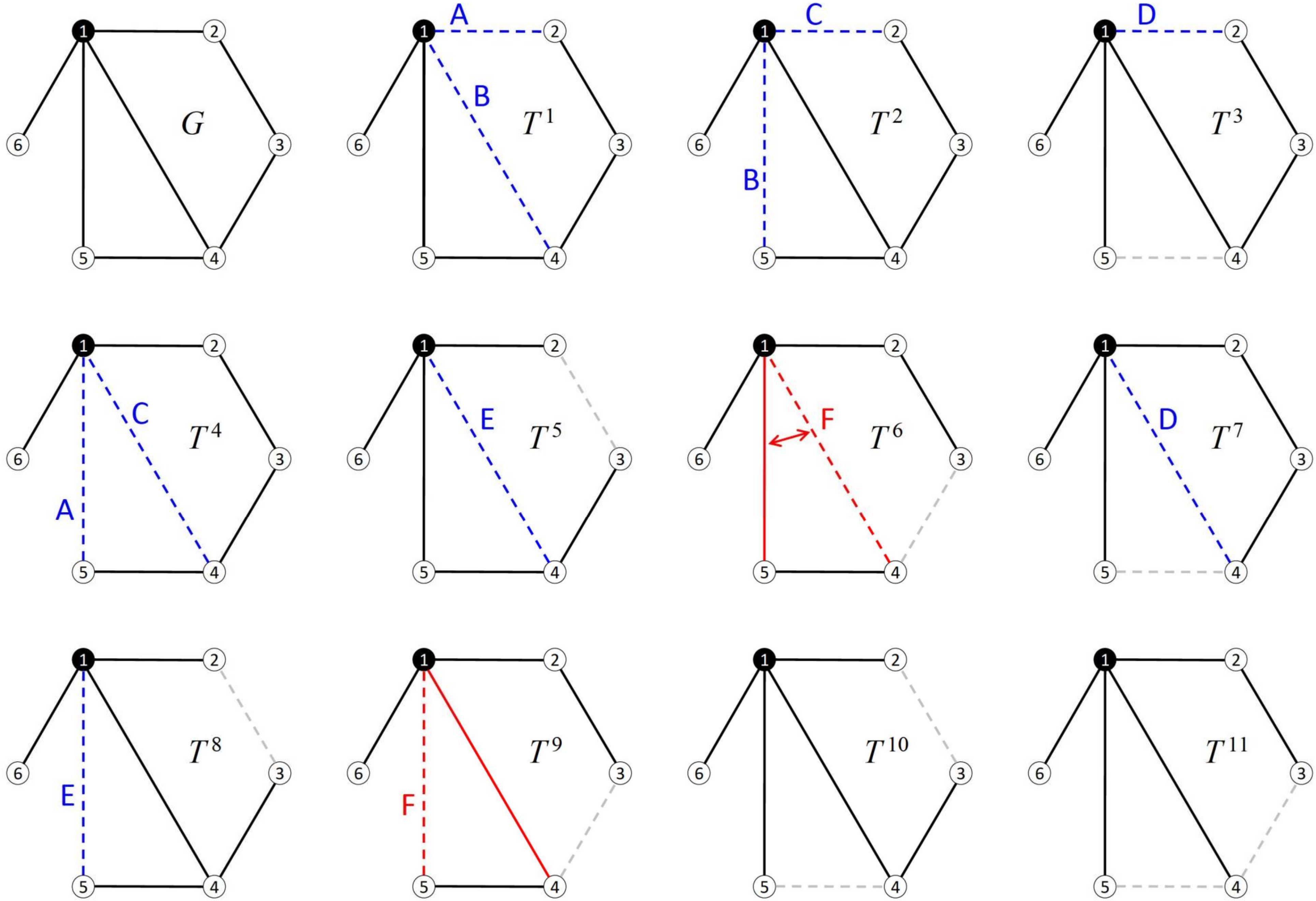

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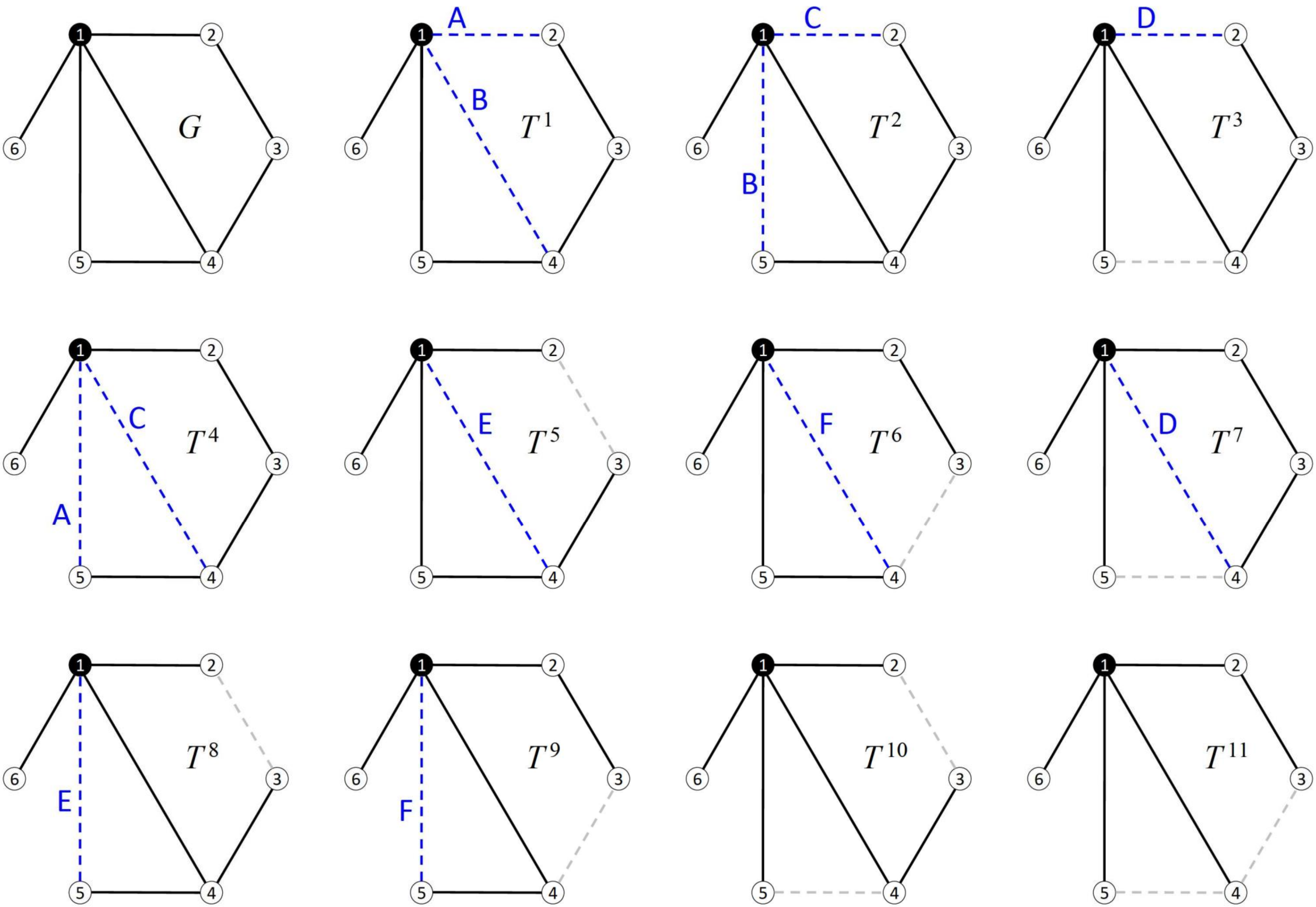

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proof

Finally, to complete the proof, take the sum of equations

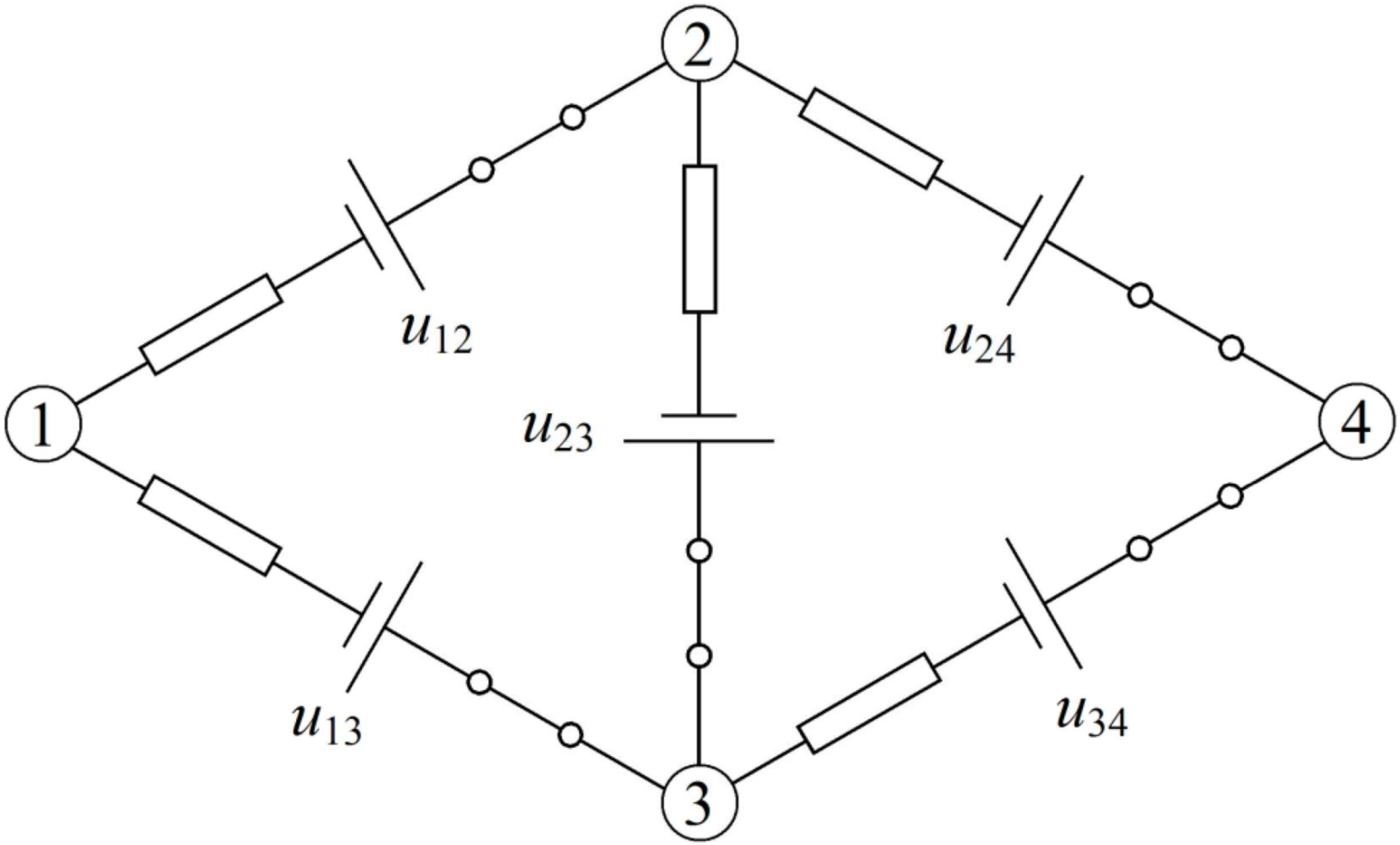
$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

for all $s = 1, 2, \ldots, S$ and apply the lemma

$$\sum_{s=1}^{S} \left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

to conclude that $\mathbf{y}^{LLS} = \frac{1}{S} \sum_{s=1}^{S} \mathbf{y}^s$. $\square$

Remark. Complete pairwise comparison matrices ($S = n^{n-2}$) are included in our theorem as a special case, and our proof can also be considered as a second, and shorter proof of the theorem of Lundy, Siraj and Greco (2017).



2
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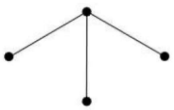
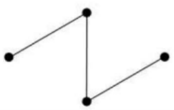
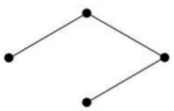
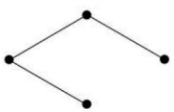
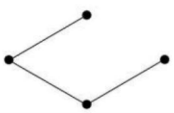
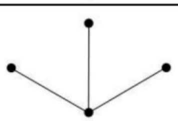
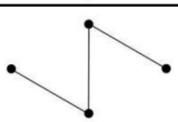
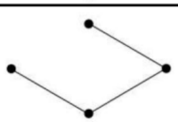
u_{12}
u_{24}
u_{23}
u_{13}
u_{34}

$$\min \sum_{\substack{i,j: \\ e(i,j) \in G}} (u_{ij} - U_i + U_j)^2$$

subject to $\quad U_1 = 0.$

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{5}{8}u_{12} + \frac{3}{8}u_{13} - \frac{1}{4}u_{23} - \frac{1}{8}u_{24} + \frac{1}{8}u_{34} \\ \frac{3}{8}u_{12} + \frac{5}{8}u_{13} + \frac{1}{4}u_{23} + \frac{1}{8}u_{24} - \frac{1}{8}u_{34} \\ \frac{1}{2}u_{12} + \frac{1}{2}u_{13} + \frac{1}{2}u_{24} + \frac{1}{2}u_{34} \end{pmatrix} = \begin{pmatrix} u_{12} + u_{13} \\ -u_{12} + u_{23} + u_{24} \\ -u_{13} - u_{23} + u_{34} \\ -u_{24} - u_{34} \end{pmatrix}$$

spanning tree	U_1	U_2	U_3	U_4
	0	u_{12}	$u_{12}+u_{23}$	$u_{12}+u_{24}$
	0	u_{12}	$u_{12}+u_{23}$	$u_{12}+u_{23}+u_{34}$
	0	u_{12}	$u_{12}+u_{24}-u_{34}$	$u_{12}+u_{24}$
	0	u_{12}	u_{13}	$u_{12}+u_{24}$
	0	u_{12}	u_{13}	$u_{13}+u_{34}$
	0	$u_{13}-u_{23}$	u_{13}	$u_{13}+u_{34}$
	0	$u_{13}-u_{23}$	u_{13}	$u_{13}-u_{23}+u_{24}$
	0	$u_{13}+u_{34}-u_{24}$	u_{13}	$u_{13}+u_{34}$
arithmetic mean	0	$\frac{5}{8}u_{12}+\frac{3}{8}u_{13}-\frac{1}{4}u_{23}$ $-\frac{1}{8}u_{24}+\frac{1}{8}u_{34}$	$\frac{3}{8}u_{12}+\frac{5}{8}u_{13}+\frac{1}{4}u_{23}$ $+\frac{1}{8}u_{24}-\frac{1}{8}u_{34}$	$\frac{1}{2}u_{12}+\frac{1}{2}u_{13}$ $+\frac{1}{2}u_{24}+\frac{1}{2}u_{34}$

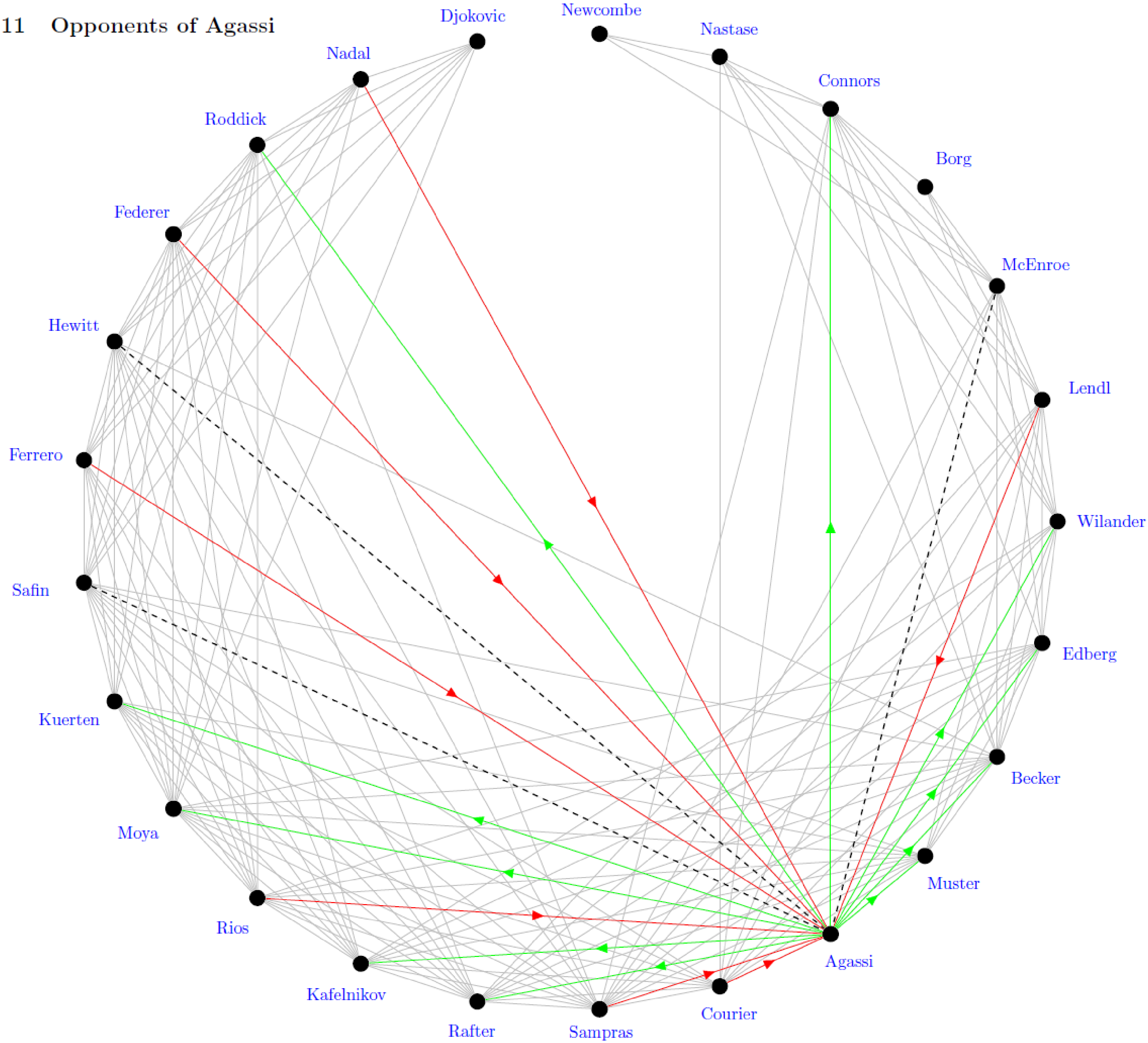
Applications of
incomplete pairwise comparison matrices

Classical multi-criteria decision models

+

Ranking
- athletes
- sport teams
- movies
- universities

1.11 Opponents of Agassi



Agassi

against	Win	Loss
Becker	10	4
Connors	2	0
Courier	5	7
Edberg	6	3
Federer	3	8
Ferrero	2	3
Hewitt	4	4
Kafelnikov	8	4
Kuerten	7	4
Lendl	2	6
McEnroe	2	2
Moya	3	1
Muster	5	4
Nadal	0	2
Rafter	10	5
Rios	1	2
Roddick	5	1
Safin	3	3
Sampras	14	20
Wilander	5	2
Sum	97	85

Pareto optimality of weight vectors

with linear programming

Inefficient principal right eigenvector

Example of Blanquero, Carrizosa and Conde (2006, p. 282):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}, \quad \mathbf{w}^* = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1.003 \\ 2.0712416 \end{pmatrix}.$$

i	a_{i3}	x_{i3}^{EM}	x_{i3}^{*}	$\lvert a_{i3} - x_{i3}^{EM}\rvert$	$\lvert a_{i3} - x_{i3}^{*}\rvert$
1	6	6.01438057	5.99639139	0.01438057	0.00360859
2	4	4.26049429	4.24775103	0.26049429	0.24775103
3	1	1	1	0	0
4	2	2.07124160	2.06504646	0.07124160	0.06504646

Definitions and notations

$\mathcal{PCM}_n$ denotes the set of pairwise comparison matrices of size $n \times n$.

$\lambda_{\max}(\mathbf{A})$ denotes the dominant eigenvalue of pairwise comparison matrix $\mathbf{A}$ of size $n \times n$.

$\mathbf{w}^{EM(\mathbf{A})}$, also called EM *weight vector*, denotes the principal right eigenvector of $\mathbf{A}$ corresponding to $\lambda_{\max}(\mathbf{A})$.

$\mathbf{w}^{EM(\mathbf{A})}$ is usually normalized to 1, that is, $\sum\limits_{i=1}^{n} w_i^{EM(\mathbf{A})} = 1$.

$\mathbf{X}^{EM(\mathbf{A})} = \mathbf{X}^{EM} \stackrel{\text{def}}{=} \left[\dfrac{w_i^{EM(\mathbf{A})}}{w_j^{EM(\mathbf{A})}} \right]_{i,j=1,\ldots,n}$ is the consistent pairwise comparison matrix generated by $\mathbf{w}^{EM(\mathbf{A})}$.
It is the approximation of $\mathbf{A}$ by the eigenvector method.

The multi-objective optimization problem is as follows:

$$\min_{x_i > 0 \; \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Efficiency

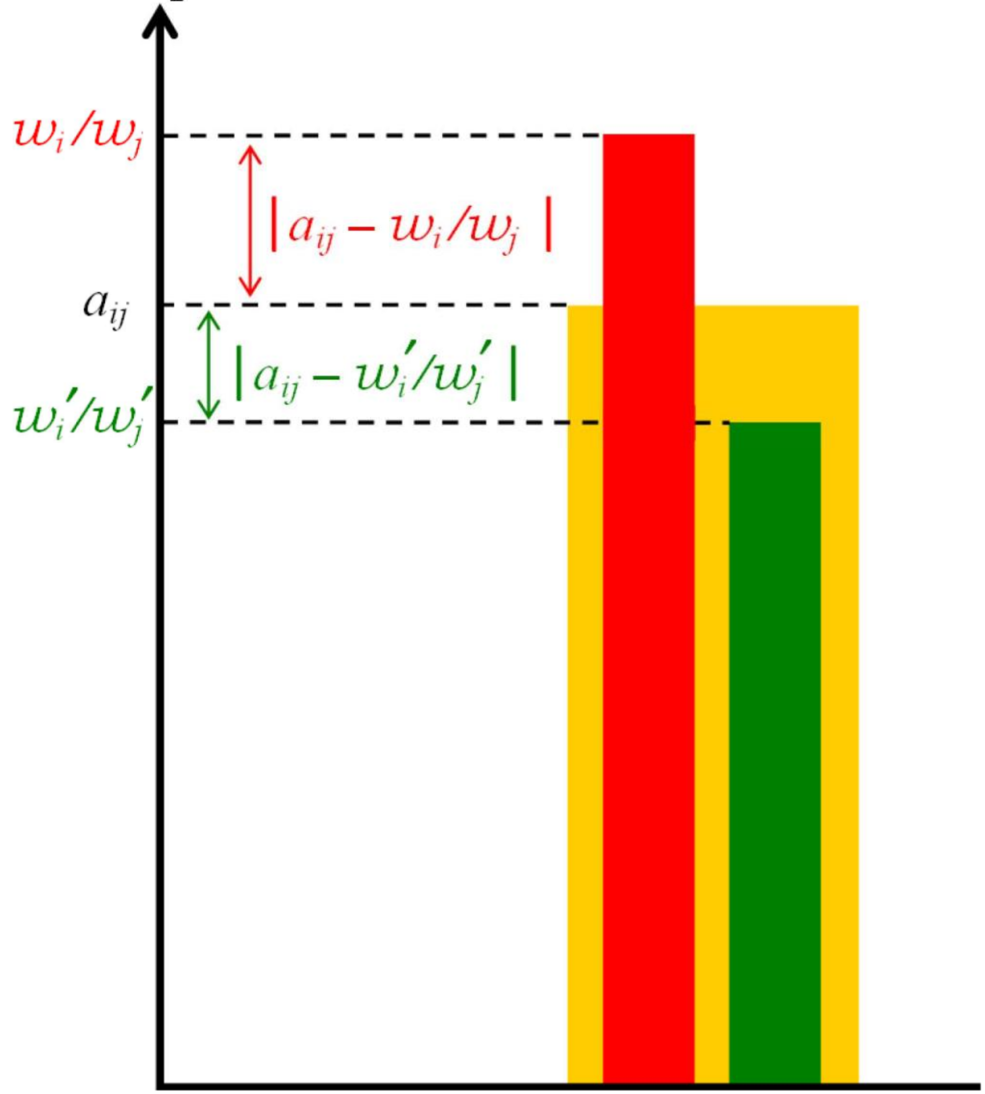
Let $\mathbf{A} = [a_{ij}]_{i,j=1,\dots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \le \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \le i, j \le n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \le k, \ell \le n.$$

An efficient weight vector cannot be improved such that every element of the pairwise comparison matrix is approximated at least as good, and at least one element is approximated strictly better.



Local efficiency

Definition: weight vector $\mathbf{w}$ is called *locally efficient,* if there is a neighborhood of $\mathbf{w}$, denoted by $V(\mathbf{w})$, such that there exists no positive weight vector $\mathbf{w}' \in V(\mathbf{w})$ fulfilling

$$\left| a_{ij} - \frac{w_i'}{w_j'} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

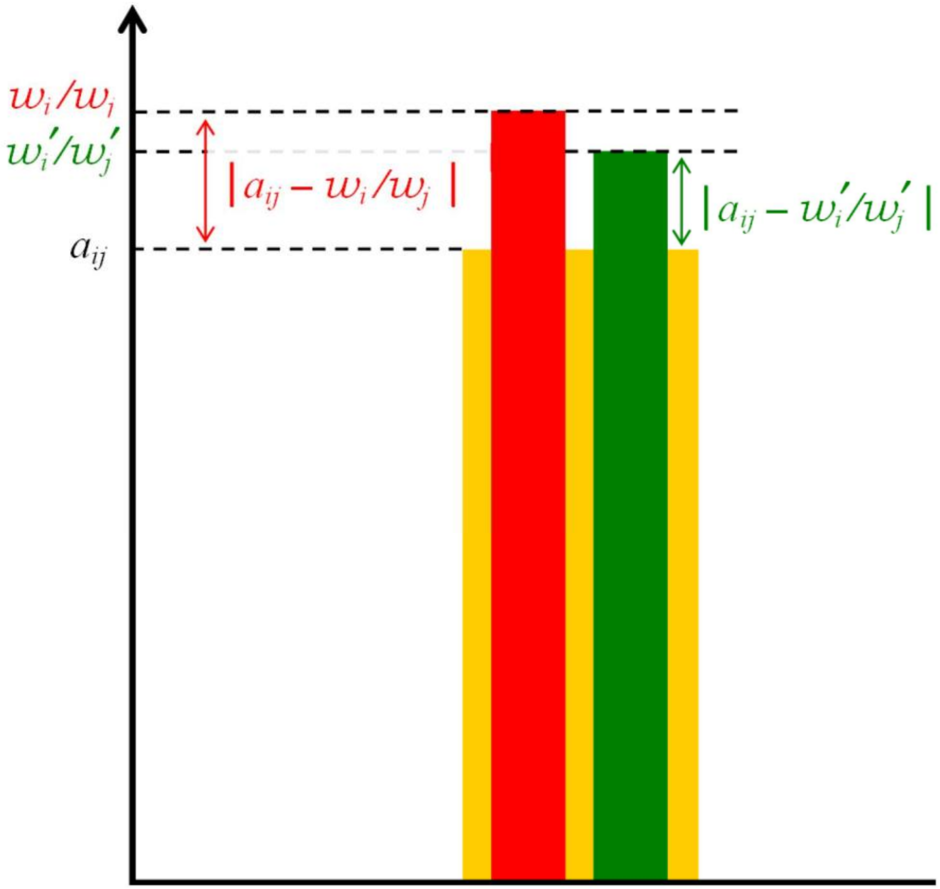
$$\left| a_{k\ell} - \frac{w_k'}{w_\ell'} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

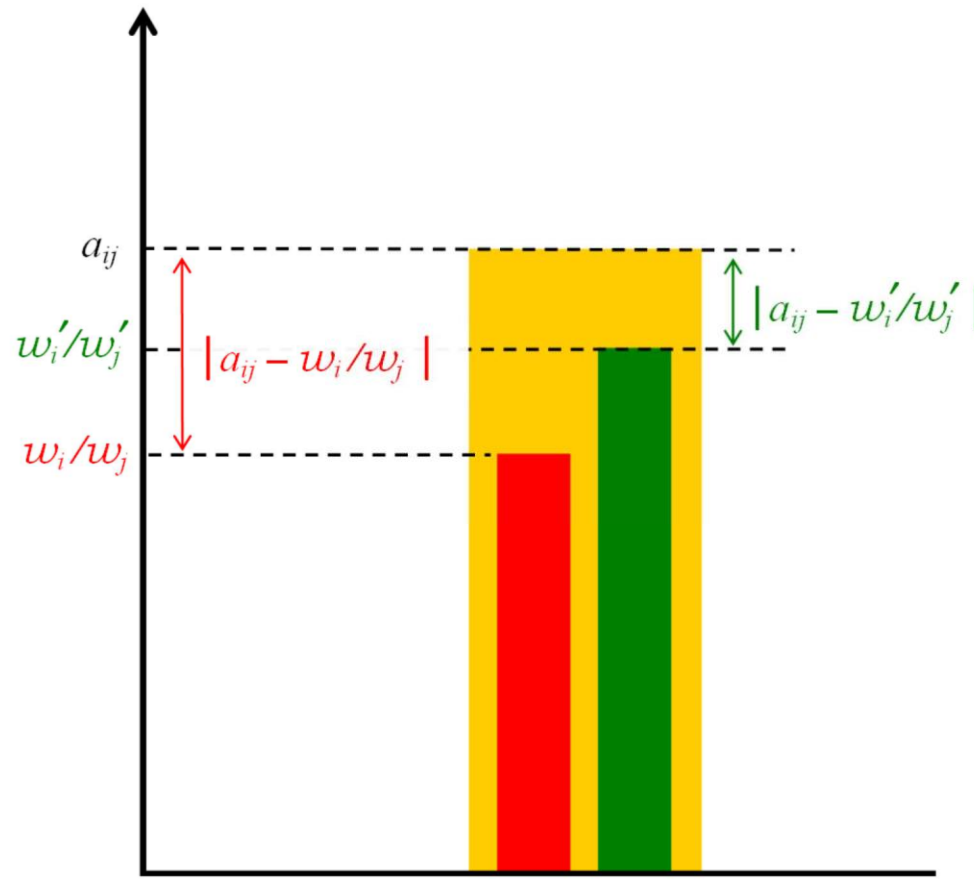
Internal efficiency

Definition: weight vector $\mathbf{w}$ is called *internally efficient,* if there exists no positive weight vector $\mathbf{w}' = (w_1', w_2', \ldots, w_n')^\top$ such that

$$\left.\begin{aligned} a_{ij} \le \frac{w_i}{w_j} \quad &\Longrightarrow \quad a_{ij} \le \frac{w_i'}{w_j'} \le \frac{w_i}{w_j} \\ a_{ij} \ge \frac{w_i}{w_j} \quad &\Longrightarrow \quad a_{ij} \ge \frac{w_i'}{w_j'} \ge \frac{w_i}{w_j} \end{aligned}\right\} \text{ for all } 1 \le i, j \le n, \text{ and}$$

$$\left.\begin{aligned} a_{k\ell} \le \frac{w_k}{w_\ell} \quad &\Longrightarrow \quad \frac{w_k'}{w_\ell'} < \frac{w_k}{w_\ell} \\ a_{k\ell} \ge \frac{w_k}{w_\ell} \quad &\Longrightarrow \quad \frac{w_k'}{w_\ell'} > \frac{w_k}{w_\ell} \end{aligned}\right\} \quad \text{for some } 1 \le k, \ell \le n.$$


w_i/w_j
w_i'/w_j'
a_{ij}
$|a_{ij} - w_i/w_j|$
$|a_{ij} - w_i'/w_j'|$


a_{ij}
w_i'/w_j'
w_i/w_j
$|a_{ij} - w_i'/w_j'|$
$|a_{ij} - w_i/w_j|$

efficient $=$ locally efficient $=$ internally efficient

Proposition (Blanquero, Carrizosa and Conde, 2006; Bozóki, Fülöp, 2015):

Definitions of

- efficiency
- local efficiency
- internal efficiency

are equivalent.

Following the way of Blanquero, Carrizosa and Conde (2006)

$$\min_{x_i > 0 \ \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Denote $\varepsilon_{ij} := \left| \frac{w_i}{w_j} - a_{ij} \right|$.

Proposition: $\mathbf{w}$ is efficient if and only if for any pair of indices $k, \ell = 1, 2, \ldots, n$, $k \neq \ell$, $\mathbf{w}$ is an optimal solution to the fractional optimization problem

$$\inf \left| \frac{x_k}{x_\ell} - a_{k\ell} \right|$$

$$\left| \frac{x_i}{x_j} - a_{ij} \right| \leq \varepsilon_{ij} \qquad \text{for all pairs } (i, j) \neq (k, \ell)$$

$$x_1, x_2, \ldots, x_n > 0.$$

$$\min \sum_{(i,j)\in I} -s_{ij} \tag{26}$$

$$y_j - y_i \le -b_{ij} \qquad \text{for all } (i, j) \in I, \tag{27}$$

$$y_i - y_j + s_{ij} \le v_i - v_j \qquad \text{for all } (i, j) \in I, \tag{28}$$

$$y_i - y_j = b_{ij} \qquad \text{for all } (i, j) \in J, \tag{29}$$

$$s_{ij} \ge 0 \qquad \text{for all } (i, j) \in I, \tag{30}$$

$$y_1 = 0 \tag{31}$$

Variables are $y_i, 1 \le i \le n$ and $s_{ij} \ge 0, (i, j) \in I$.

Theorem 4.1. *The optimum value of the linear program (26)–(31) is at most 0 and it is equal to 0 if and only if weight vector* $\mathbf{w}$ *is efficient for (1).*

Characterization of efficiency

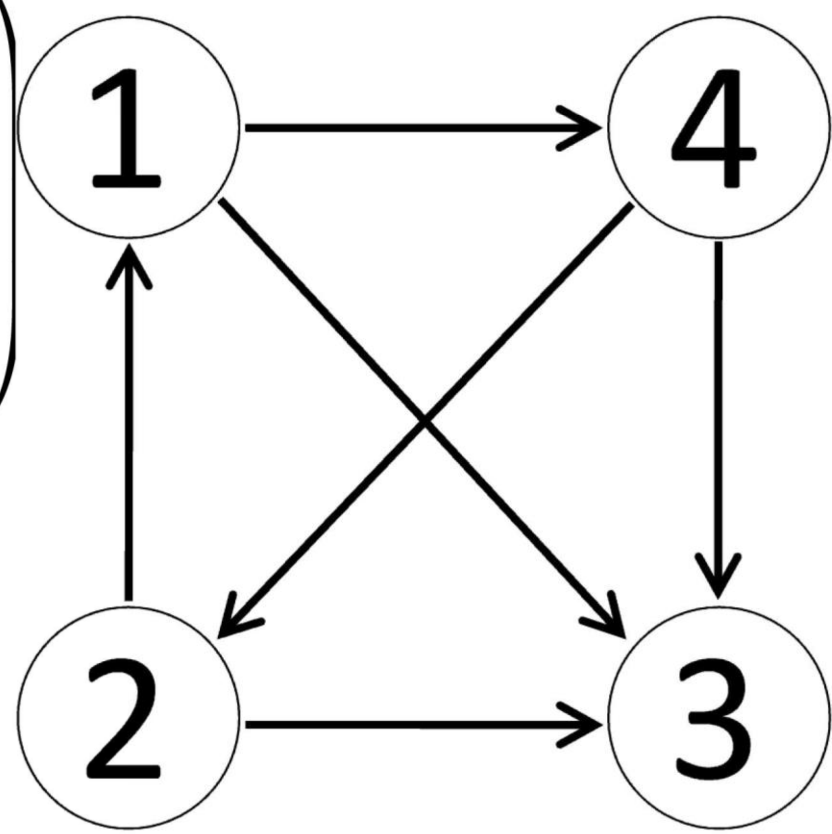
Definition: Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector. Directed graph $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is defined as follows: $V = \{1, 2, \ldots, n\}$ and

$$\overrightarrow{E} = \left\{ \text{arc}(i \rightarrow j) \left| \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right. \right\}.$$

Theorem (Blanquero, Carrizosa and Conde, 2006): Weight vector $\mathbf{w}$ is efficient if and only if $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is strongly connected, that is, there exist directed paths from i to j and from j to i for all pairs of $i \neq j$ nodes.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}$$

$$\mathbf{X}^{EM} = \begin{pmatrix} 1 & \color{blue}{1.41} & \color{red}{6.01} & \color{red}{2.90} \\ \color{red}{0.71} & 1 & \color{red}{4.26} & \color{blue}{2.06} \\ \color{blue}{0.1663} & \color{blue}{0.23} & 1 & \color{blue}{0.48} \\ \color{blue}{0.34} & \color{red}{0.49} & \color{red}{2.07} & 1 \end{pmatrix}$$



Special cases

Efficient principal right eigenvector:

- simple perturbed PCM
- double perturbed PCM

Inefficient principal right eigenvector:

- PCM with arbitrarily small inconsistency
- Numerical examples

Simple perturbed PCM

Consider a consistent matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n,$$

then perturb a single element and its reciprocal. The perturbation is realized by a multiplication by $\delta > 0, \delta \neq 1$, while the reciprocal element is divided by δ.

Simple perturbed PCM: $\mathbf{w}^{EM}$ is efficient

$$\mathbf{A}_\delta = \begin{pmatrix} 1 & \delta x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n.$$

Theorem (Ábele-Nagy, Bozóki, 2015):
The principal right eigenvector of a simple perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$.

Double perturbed PCM ($n \geq 4$)

$$\begin{pmatrix} 1 & \delta x_1 & \gamma x_2 & x_3 & \cdots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{\gamma x_2} & \frac{x_1}{x_2} & 1 & \frac{x_3}{x_2} & \cdots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{x_3} & 1 & \cdots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \delta x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \gamma \frac{x_3}{x_2} & \cdots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{\gamma x_3} & 1 & \cdots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \cdots & 1 \end{pmatrix}$$

Double perturbed PCM: $\mathbf{w}^{EM}$ is efficient

Theorem (Ábele-Nagy, Bozóki, Rebák, 2015):
The principal right eigenvector of a double perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$ and the characterization of efficiency by a strongly connected digraph.

$$\mathbf{A}(p,q) = \begin{pmatrix} 1 & p & p & p & \dots & p & p \\ 1/p & 1 & q & 1 & \dots & 1 & 1/q \\ 1/p & 1/q & 1 & q & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1/p & 1 & 1 & 1 & \dots & 1 & q \\ 1/p & q & 1 & 1 & \dots & 1/q & 1 \end{pmatrix},$$

Proposition. (Bozóki, 2014) Let q be positive and $q \neq 1$. Then $\mathbf{w}^{EM}$ is internally inefficient, therefore inefficient. Furthermore, CR inconsistency can be arbitrarily small if q is close enough to 1.

Pareto optimality of weight vectors

- eigenvector
- arithmetic mean of spanning trees' weight vectors
- cosine

Weighting methods

cosine maximization – similarity of vectors measured by their angle

AMAST - arithmetic mean of weight vectors calculated from all spanning trees (minimally sufficient subset of pairwise comparisons).

Geometric mean is another possibility, it is equivalent to the logarithmic least squares method.

Efficiency (Pareto optimality)

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

An efficient weight vector cannot be improved such that every element of the pairwise comparison matrix is approximated at least as good, and at least one element is approximated strictly better.

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \ \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \; \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \ \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \ \mathbf{w}^{*} = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \ \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \ \mathbf{w}^* = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\left[\frac{w_i'}{w_j'}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}.$$

Main questions

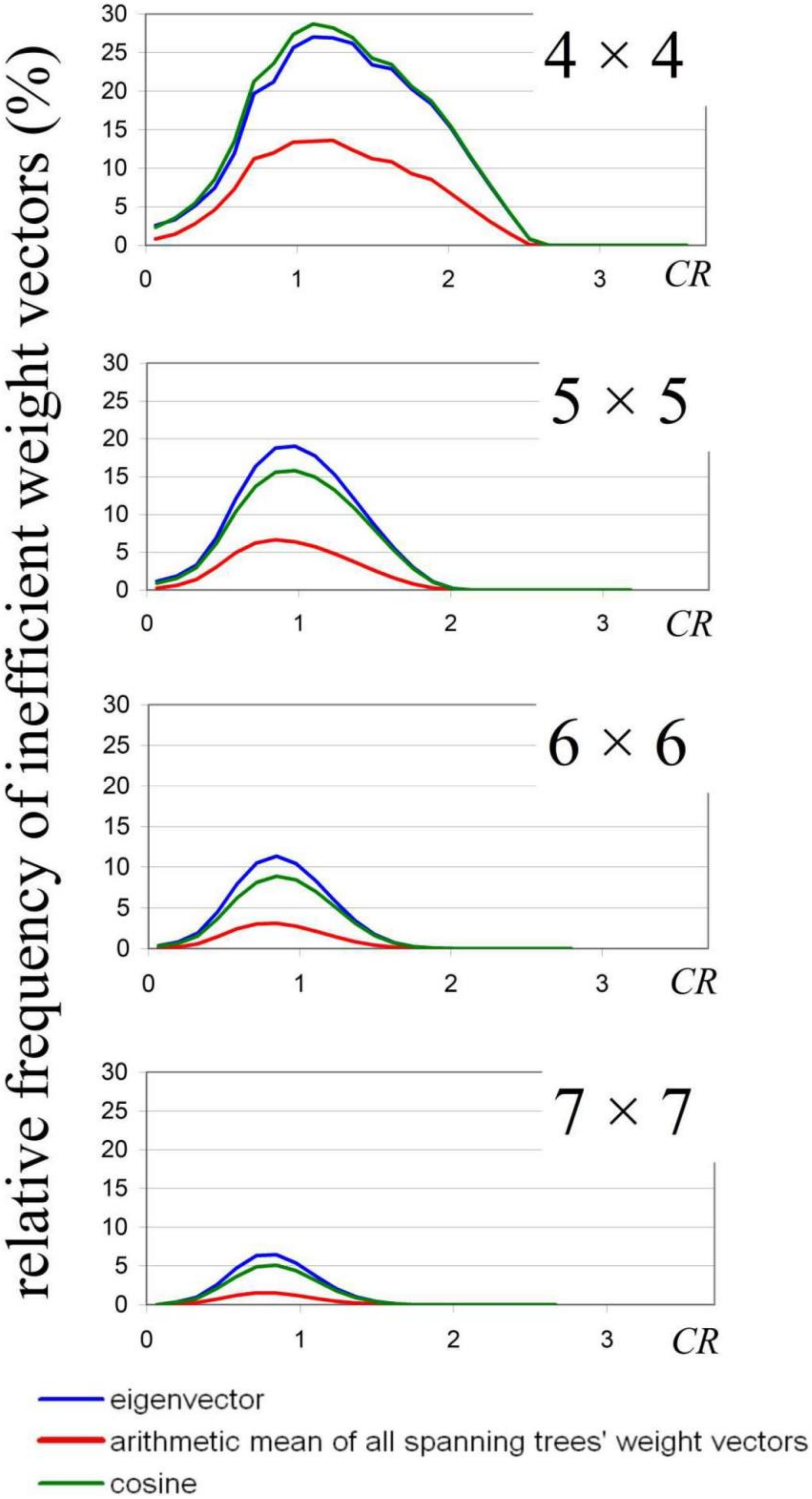
How often are the eigenvector, the AMAST and the cosine weight vectors inefficient?

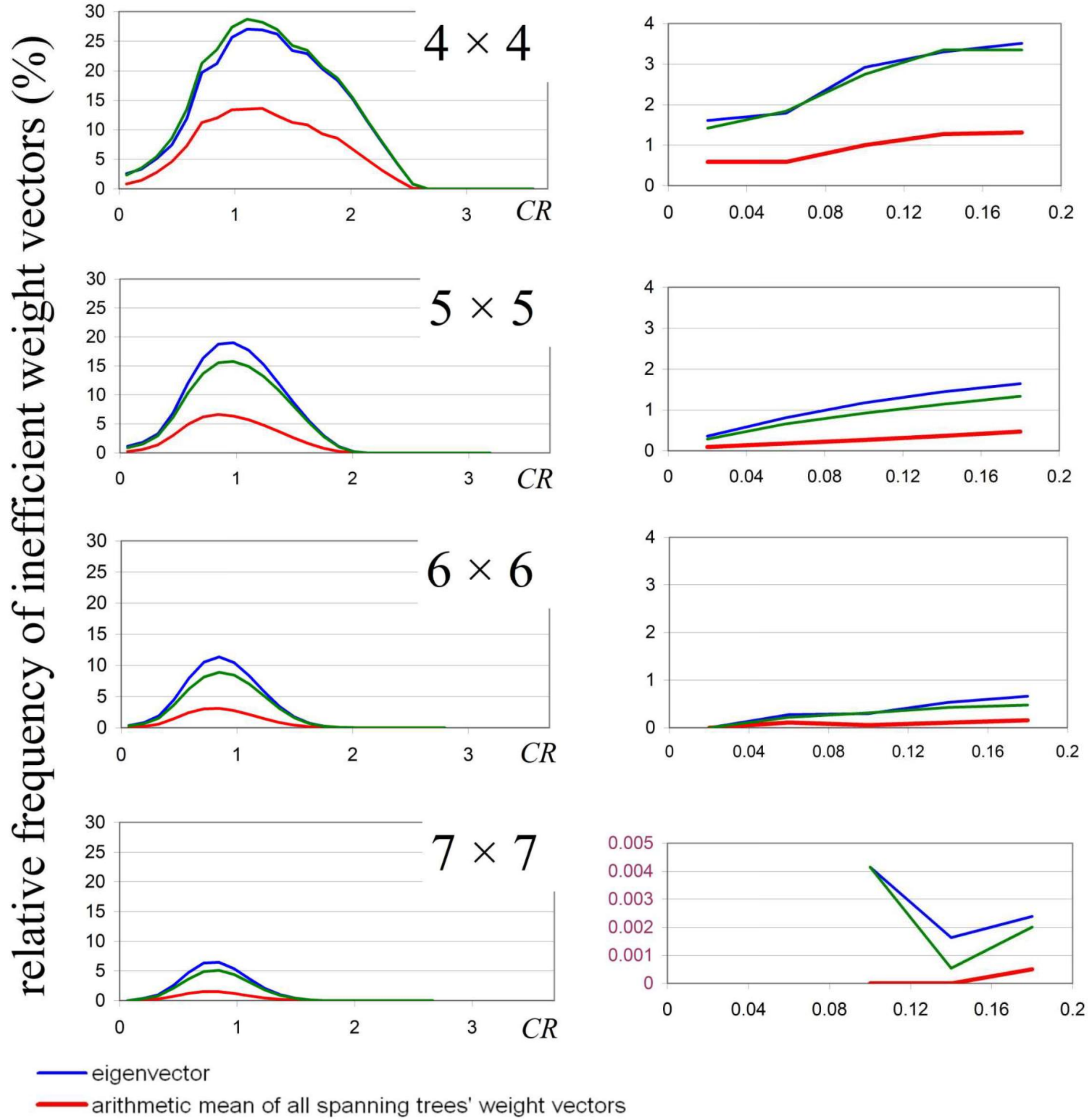
Why are the eigenvector, the AMAST and the cosine weight vectors inefficient? Can we give necessary and sufficient conditions?

4×4 matrices

There are $1\,007\,097$ pairwise comparison matrices of size 4×4 such that all elements are from the ratio scale $1, 2, \ldots, 9, 1/2, 1/3, \ldots, 1/9$, and no pair of matrices can be transformed into each other by row/column permutations (without permutation filtering there would be $17^6 = 24\,137\,569$ matrices).
Out of the $32\,157$ permutation filtered matrices fulfilling $CR \leq 0.1$ (CR is an inconsistency index),

- 591 (1.84%) have inefficient eigenvector
- 197 (0.61%) have inefficient weight vector calculated by the spanning trees' arithmetic mean (AMAST)
- 602 (1.87%) have inefficient weight vector calculated by the cosine maximization method


relative frequency of inefficient weight vectors (%)
4 × 4
5 × 5
6 × 6
7 × 7
CR
0
1
2
3
0
5
10
15
20
25
30
eigenvector
arithmetic mean of all spanning trees' weight vectors
cosine



relative frequency of inefficient weight vectors (%)
4 × 4
5 × 5
6 × 6
7 × 7
CR
eigenvector
arithmetic mean of all spanning trees' weight vectors
cosine

Main questions

How often are the eigenvector, the AMAST and the cosine weight vectors inefficient?

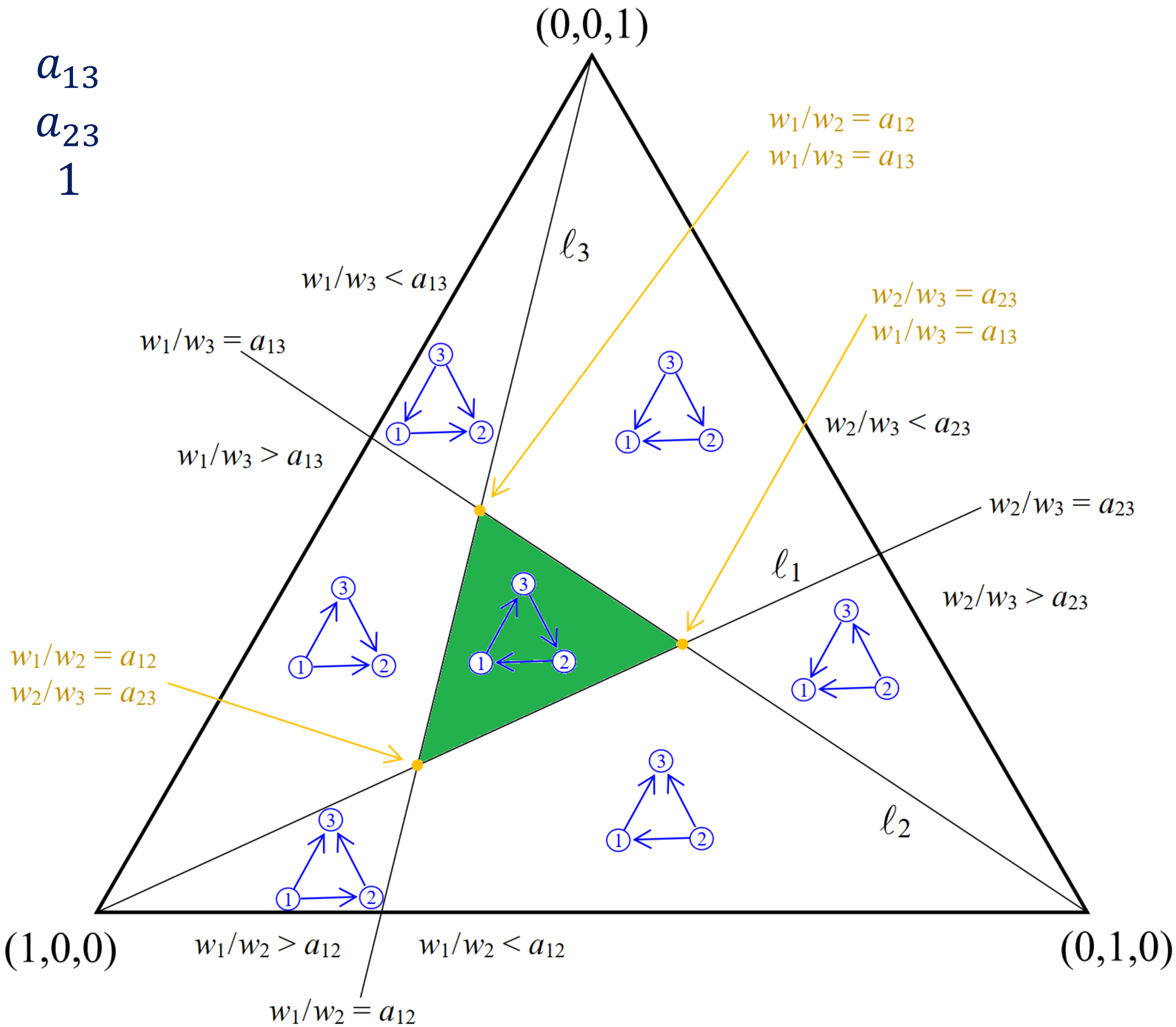
Not too often, but not at all with negligible frequency.
For small CR, the higher CR is, the more frequent inefficiency is.

Why are the eigenvector, the AMAST and the cosine weight vectors inefficient? Can we give necessary and sufficient conditions?

Not yet, it is open for all the three methods.

Geometry of efficient weight vectors

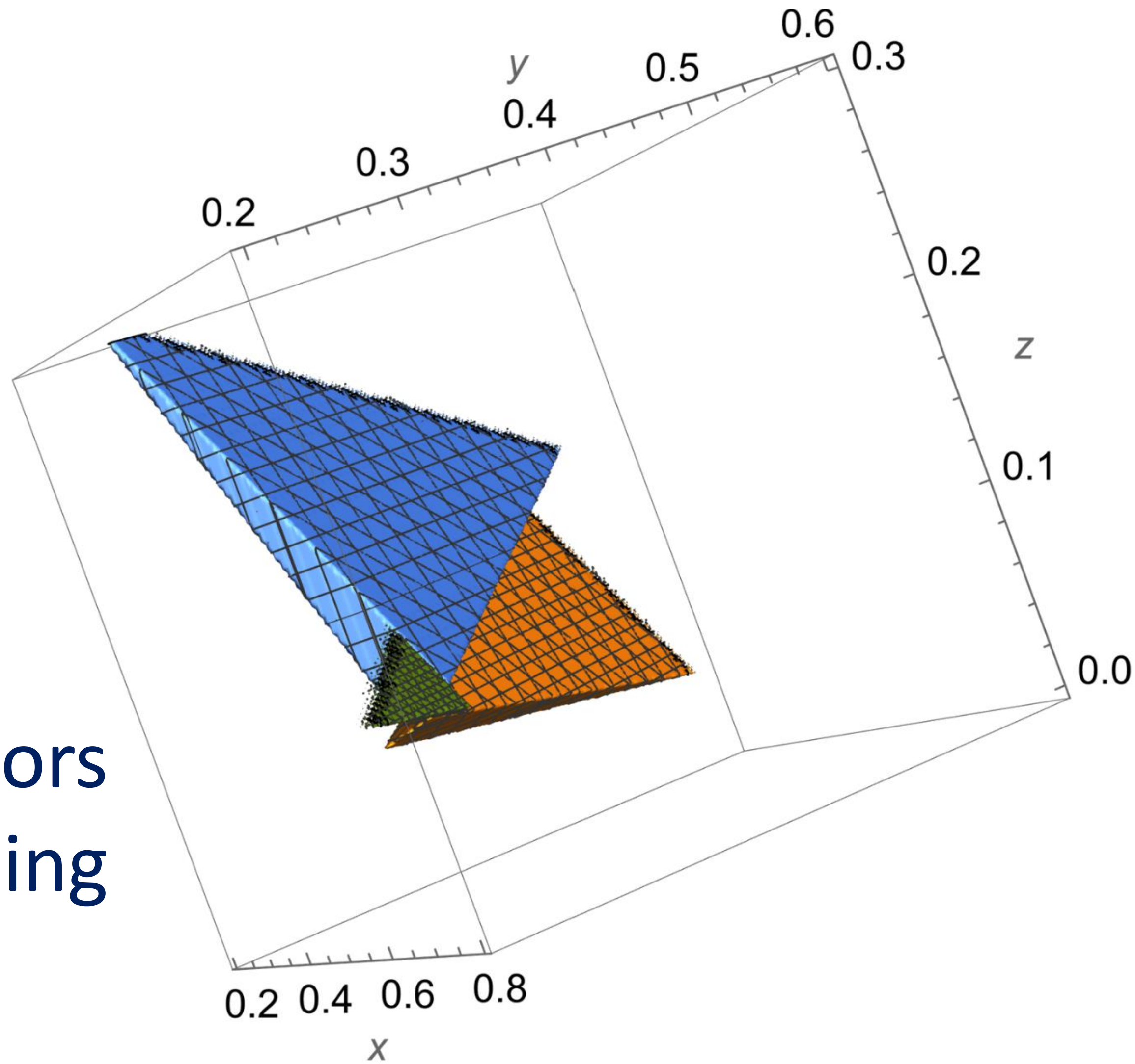
$$\begin{matrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{matrix}$$



4×4 matrix

union of three tetrahedron

Each tetrahedron is spanned by weight vectors corresponding to spanning trees



GRAPH of unlabelled graphs

Our assumptions

- We can choose the comparisons, i.e., they are not given a priori.
- We do not have any further prior information, we treat the items to be compared in a symmetric way (the isomorphic representing graphs are not distinguished).

Research question

Research question

For a given number of alternatives (n) and a given number of comparisons (e) which filling in pattern estimates the preferences of the decision maker in the best way from all the possible patterns? What is the relation between these filling in patterns for different values of e?

In this study we only focus on connected representing graphs and the cases with $n \leq 6$.

Simulation

- Weight calculation techniques: Logarithmic Least Squares Method (**LLSM**) (and the **Eigenvector Method** based on the CR-minimal completion)
- Metrics: **Euclidean distance** and **Kendall's tau**:

$$d_{euc}(u, v) = \sqrt{\sum_{i=1}^{n}(u_i - v_i)^2} \tag{1}$$

$$\tau(u, v) = \frac{n_c(u, v) - n_d(u, v)}{n(n-1)/2} \tag{2}$$

- We always compare the results to the weightvector calculated from the complete case, the applied sample-size is 1 000 000
- $\alpha = 0.01$ significance level and $\epsilon = 0.0005$ margin of error in case of the Euclidean distance, $\alpha = 0.05$ significance level and $\epsilon = 0.001$ margin of error in case of the Kendall's τ

Simulation II

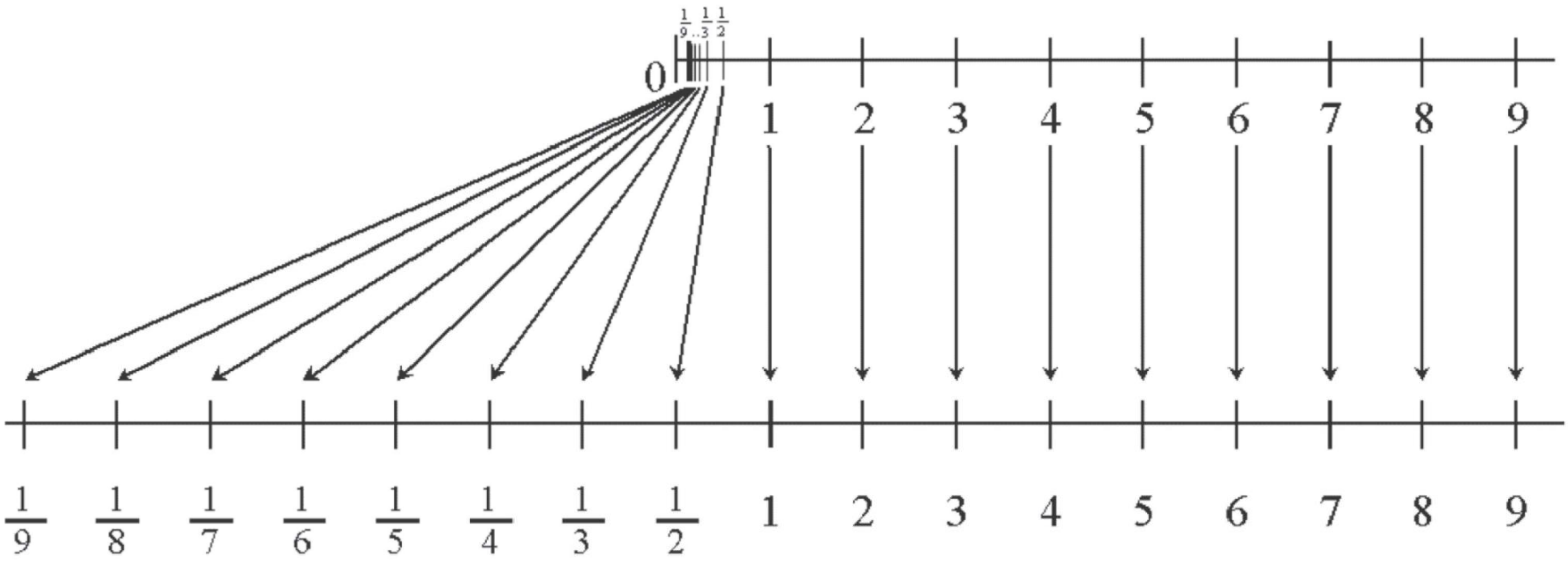
- We use three well-distinguishable inconsistency (perturbation) levels

$$\hat{a}_{ij}^{weak} = \begin{cases} a_{ij} + \Delta_{ij} & : a_{ij} + \Delta_{ij} \geq 1 \\ \frac{1}{1-\Delta_{ij}-(a_{ij}-1)} & : a_{ij} + \Delta_{ij} < 1 \end{cases} \qquad \Delta_{ij} \in [-1, 1] \quad (3)$$

$$\hat{a}_{ij}^{modest} = \begin{cases} a_{ij} + \Delta_{ij} & : a_{ij} + \Delta_{ij} \geq 1 \\ \frac{1}{1-\Delta_{ij}-(a_{ij}-1)} & : a_{ij} + \Delta_{ij} < 1 \end{cases} \qquad \Delta_{ij} \in \left[-\frac{3}{2}, \frac{3}{2}\right] \quad (4)$$

$$\hat{a}_{ij}^{strong} = \begin{cases} a_{ij} + \Delta_{ij} & : a_{ij} + \Delta_{ij} \geq 1 \\ \frac{1}{1-\Delta_{ij}-(a_{ij}-1)} & : a_{ij} + \Delta_{ij} < 1 \end{cases} \qquad \Delta_{ij} \in [-2, 2] \quad (5)$$

Simulation III



$$a_{ij} = 2, \quad \Delta_{ij} = -2$$

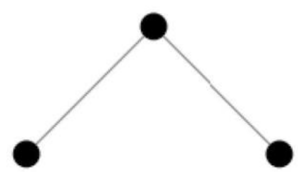
$$\hat{a}_{ij}^{strong} = 1/(1+2-1) = 1/2$$

Results

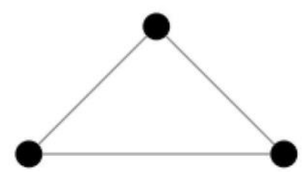
- There is a strong relation between the different measures and the different perturbation levels
- The filling in patterns (the representing graphs) can be ranked, thus we can determine the best one
- The problem is only interesting from $n = 4$
- We present our results with the help of the GRAPH of graphs concept

$n = 2, e = 1$

$n = 3, e = 2$



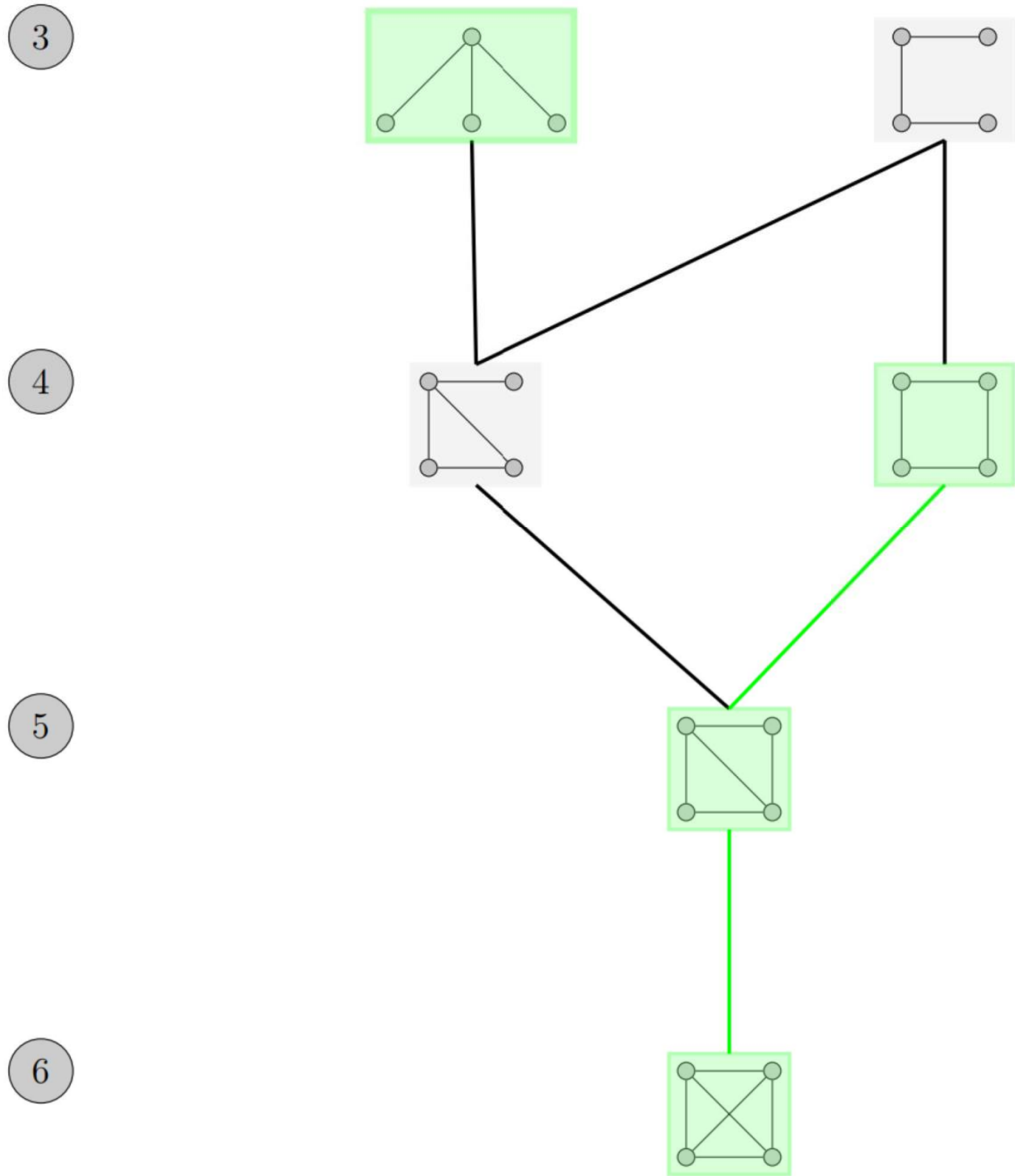
$n = 3, e = 3$



Other GRAPHs of graphs

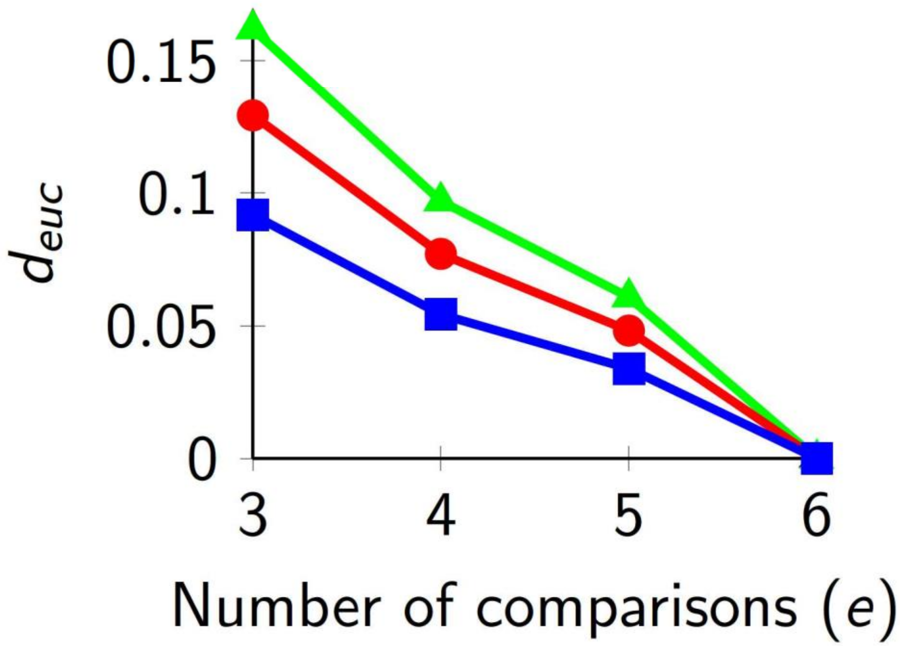
- Bondy and Lovász (1977) showed that the GRAPH of graphs is connected, where GRAPH is defined as follows: let G be a 2-connected graph on n nodes, v is a node of G; NODEs are the spanning trees of G, and two NODEs are connected by an EDGE if the corresponding spanning trees have a common subtree on $n-1$ nodes including v.
- The GRAPH of graphs of the Petersen family of seven graphs, including the Petersen graph itself (see Hashimoto and Nikkuni (2013)). Two graphs are connected by an EDGE if they can be transformed from each other by replacing a triangle by a 3-star (including the addition of its center).
- The GRAPH of graphs by Mesbahi (2002) is motivated by the evolution of graphs in a dynamic system.

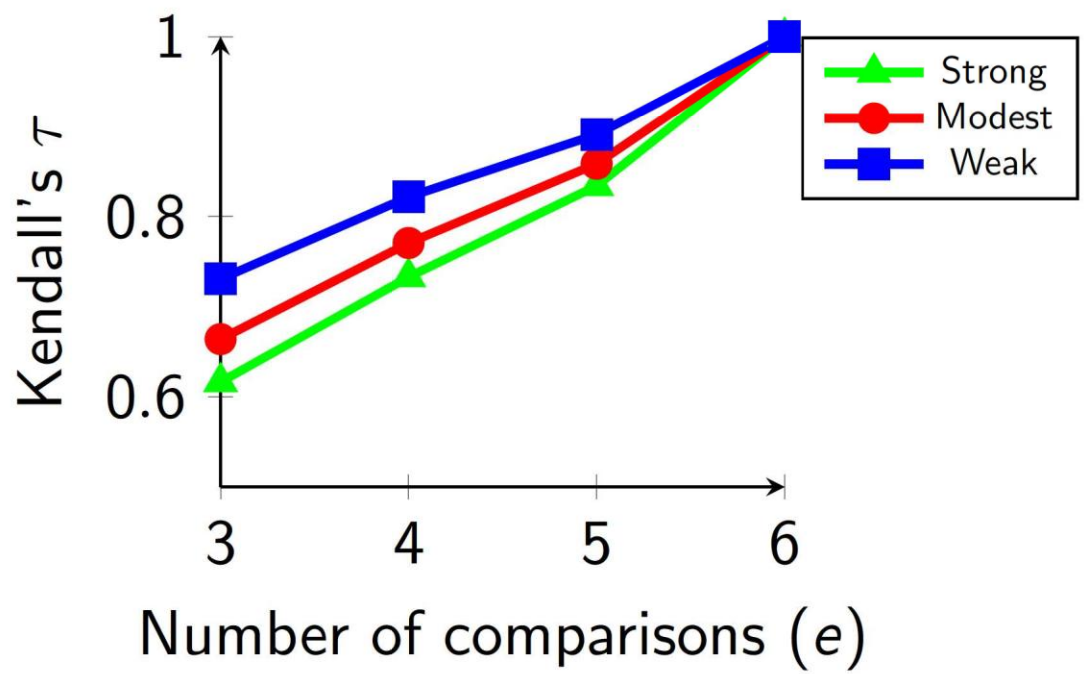
Results: n=4 GRAPH of graphs



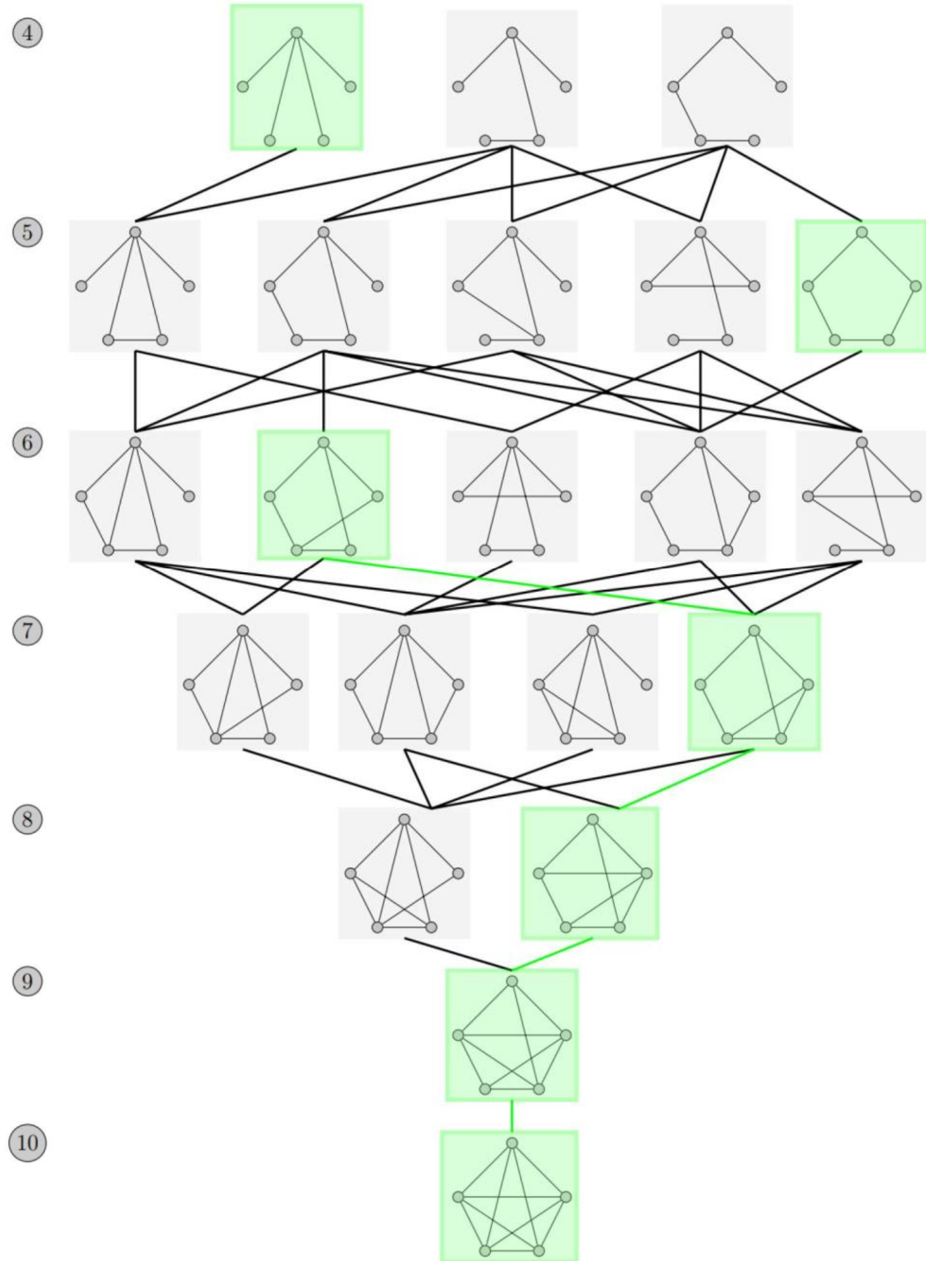
Results: n=4

	1	2	3	4
1		#1'	#5	#4'
2			#2'	#6
3				#3'
4				



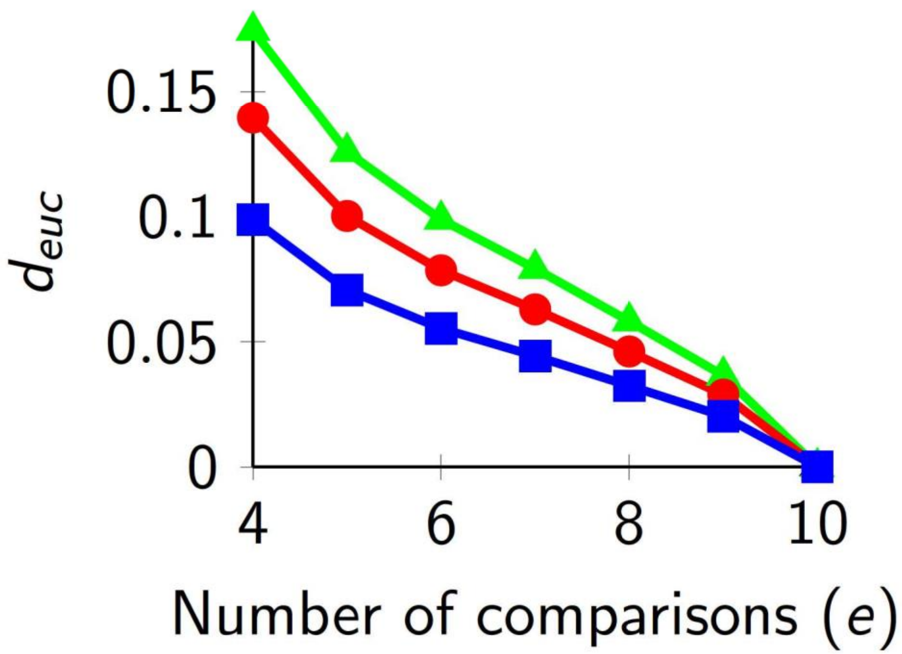


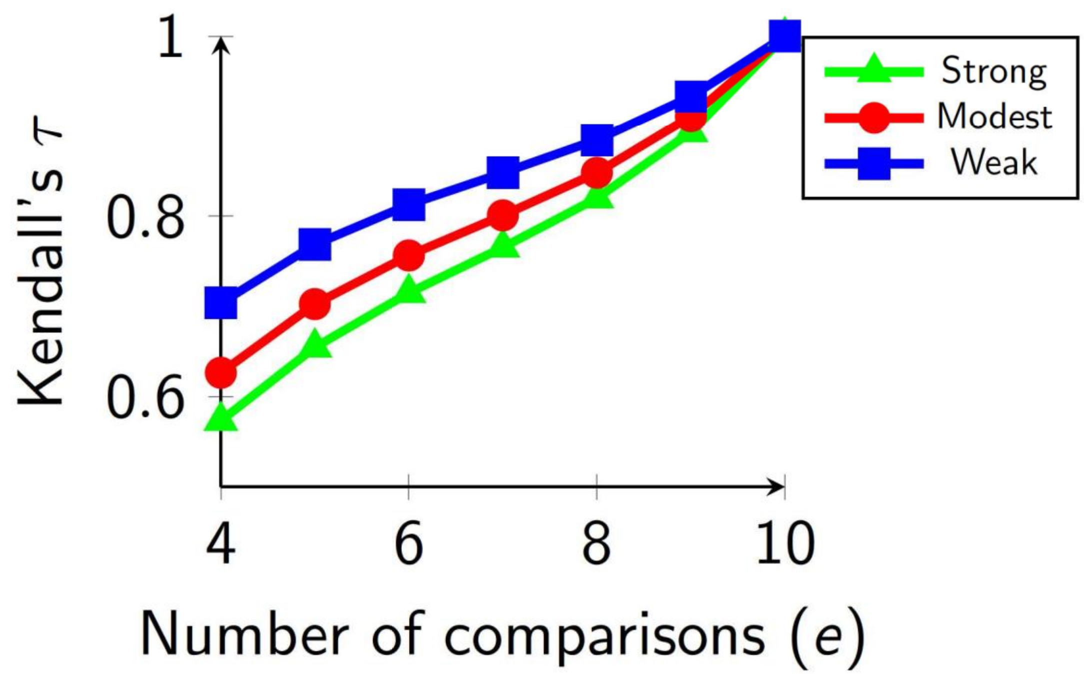
Results: n=5 GRAPH of graphs



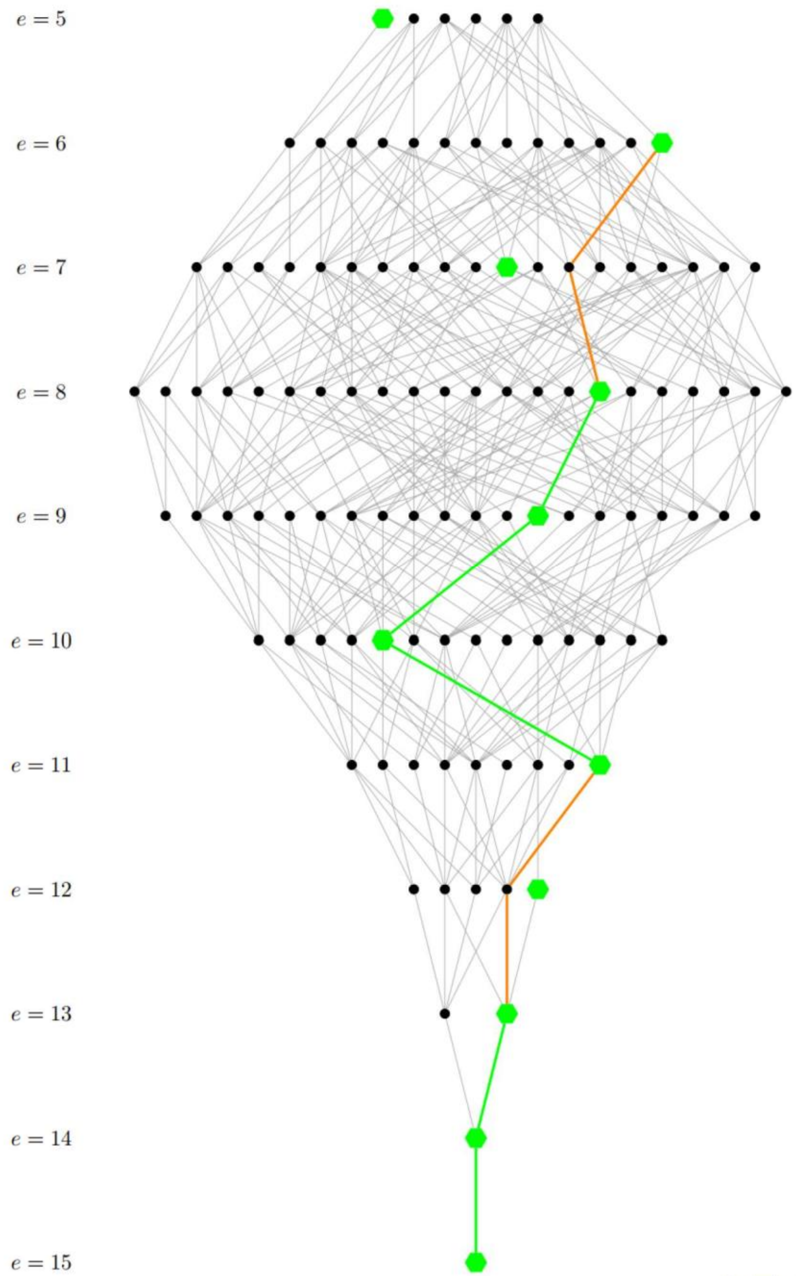
Results: n=5

	1	2	3	4	5
1		#7	#8	#1'	#2'
2			#10	#3'	#4'
3				#5'	#6'
4					#9
5					



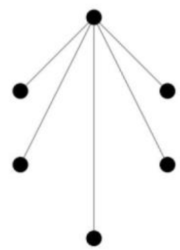


Results: n=6 GRAPH of graphs

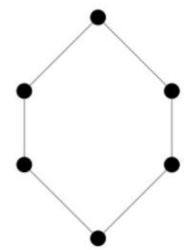


Results: n=6 optimal graphs

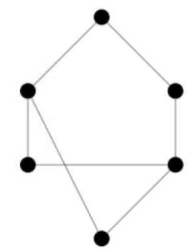
$n = 6, e = 5$



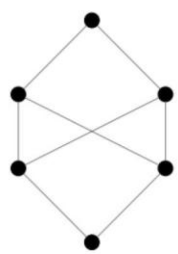
$n = 6, e = 6$



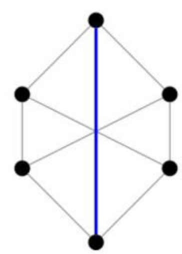
$n = 6, e = 7$



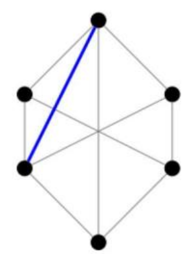
$n = 6, e = 8$



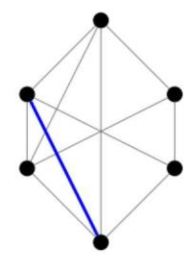
$n = 6, e = 9$



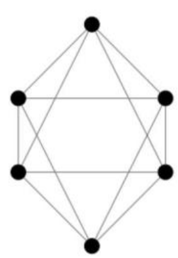
$n = 6, e = 10$



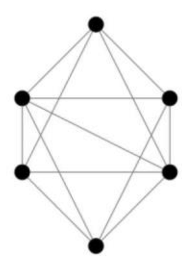
$n = 6, e = 11$



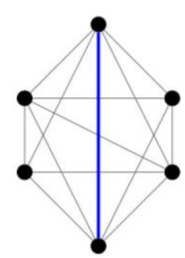
$n = 6, e = 12$



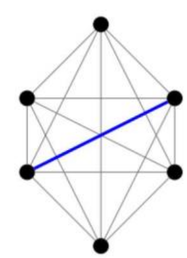
$n = 6, e = 13$



$n = 6, e = 14$

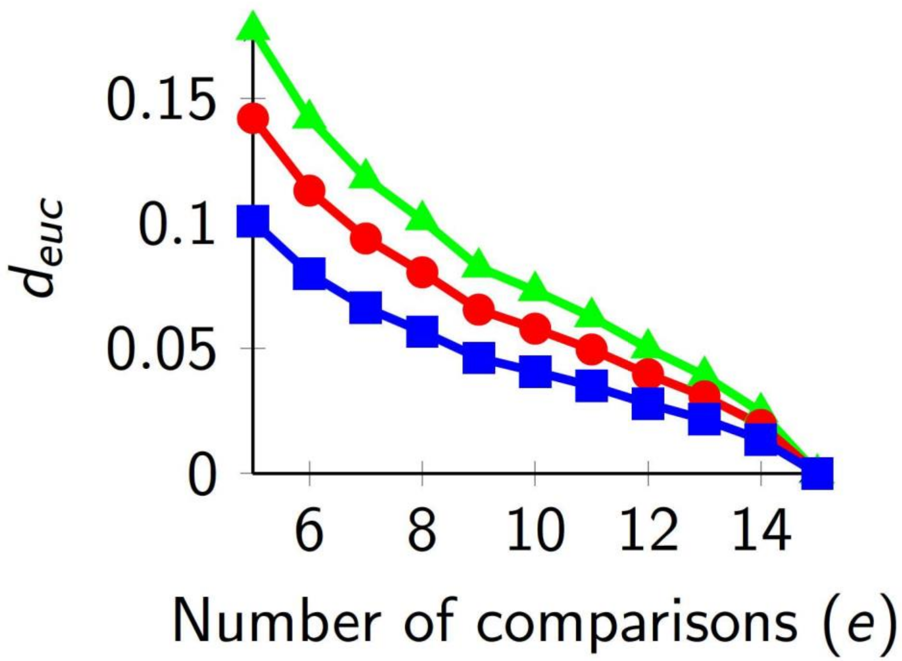


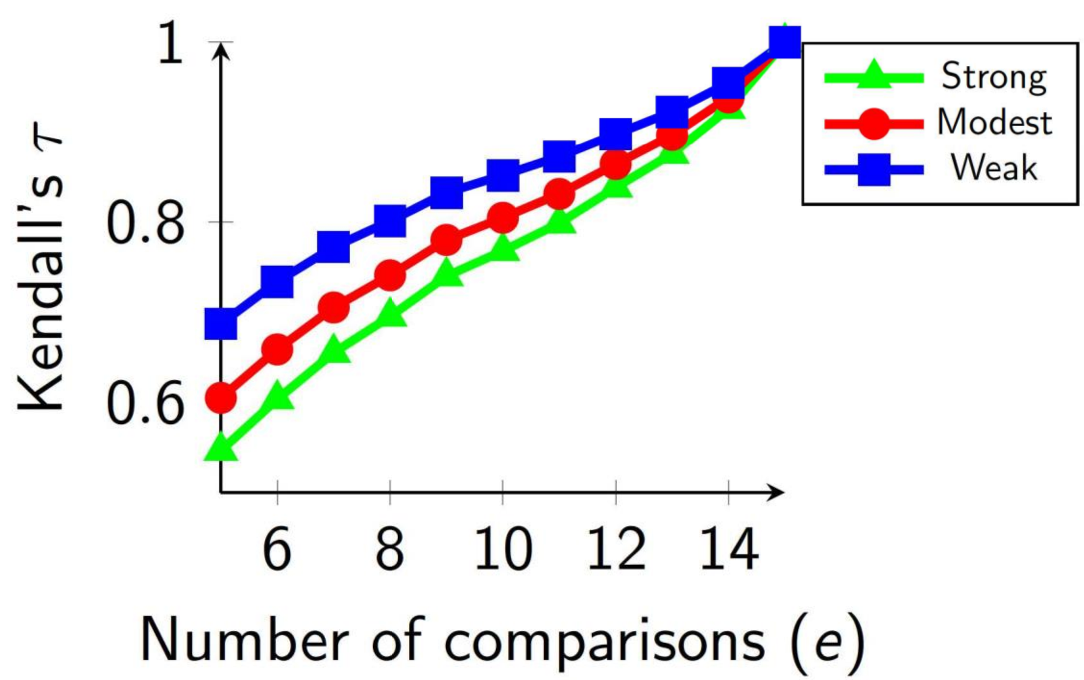
$n = 6, e = 15$



Results: n=6

	1	2	3	4	5	6
1		#10	#15	#1'	#2'	#9
2			#12	#3'	#7	#4'
3				#8	#5'	#6'
4					#13	#11
5						#14
6						





Summary

- We examined filling in patterns of incomplete pairwise comparison matrices for all possible patterns in case of a given number of alternatives and a given number of comparisons up until 6 alternatives.
- We determined the optimal graphs based on the simulations, and in many cases we were able to find optimal filling in sequences as well, which were presented with the help of the concept of GRAPH of graphs.
- The proposed filling in patterns and sequences can be used easily and rapidly in practical problems.

Future research

- Which comparisons have the largest impact on the results? At which level of margin of error should we stop the filling in process?
- Optimal filling in sequences for larger cases, indirect paths between graphs (inclusion relations) and adding ordinal information to the study could also be interesting.
- We would like to create a collection of different GRAPH of graphs.

Key references

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- Lovász, L (1977). A homology theory for spanning trees of a graph. Acta Mathematica Academiae Scientiarum Hungaricae, 30(3-4):241-251.

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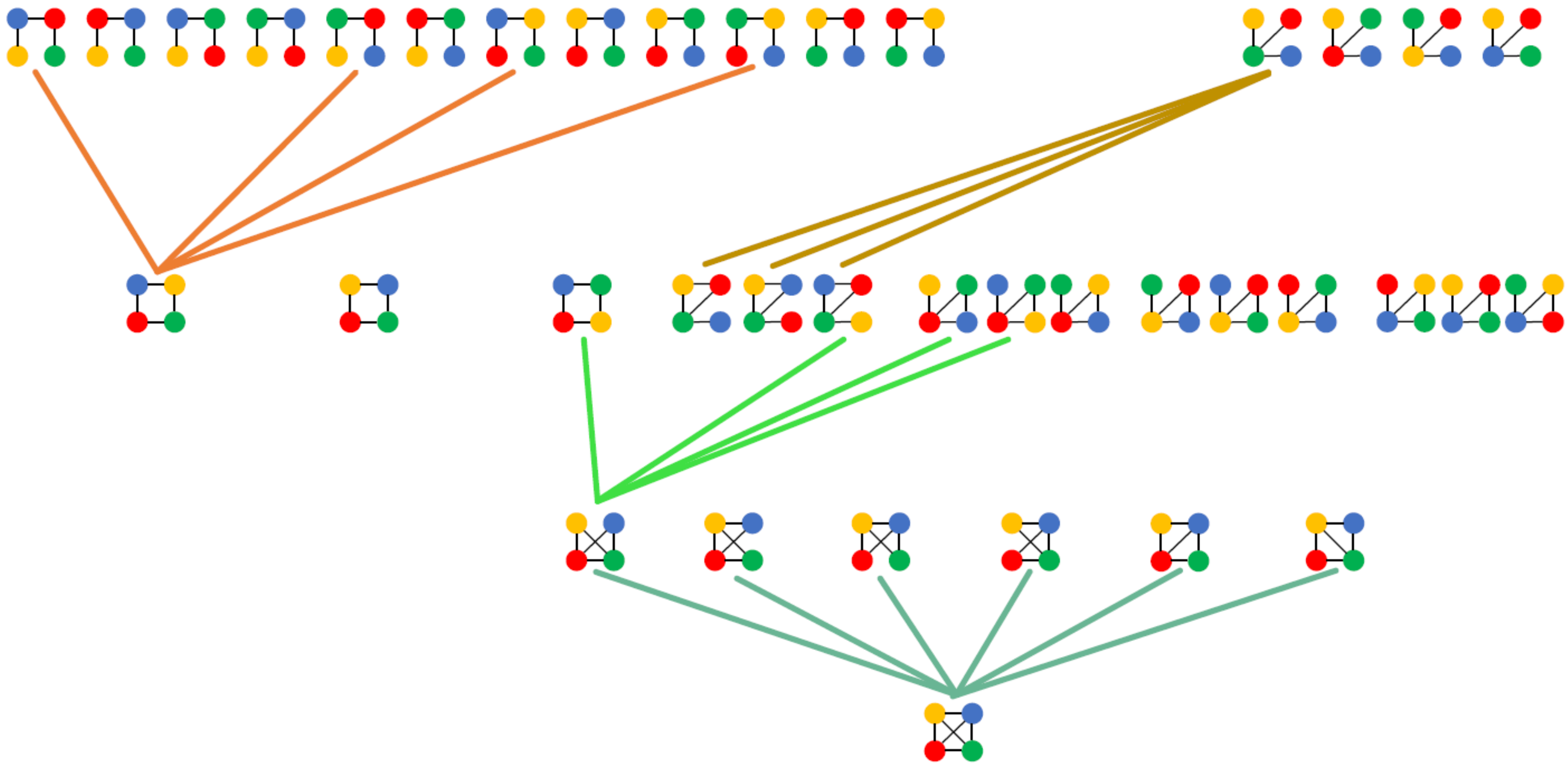
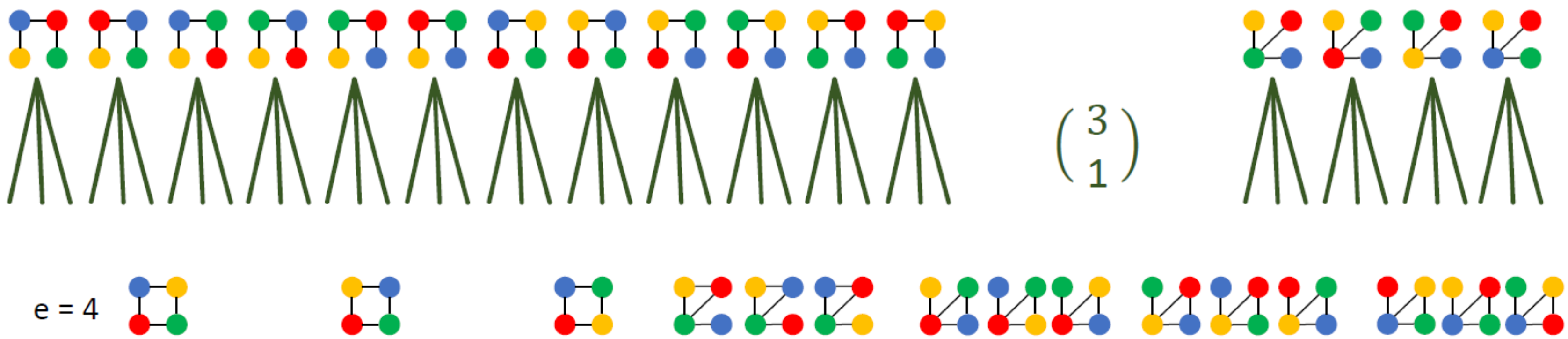
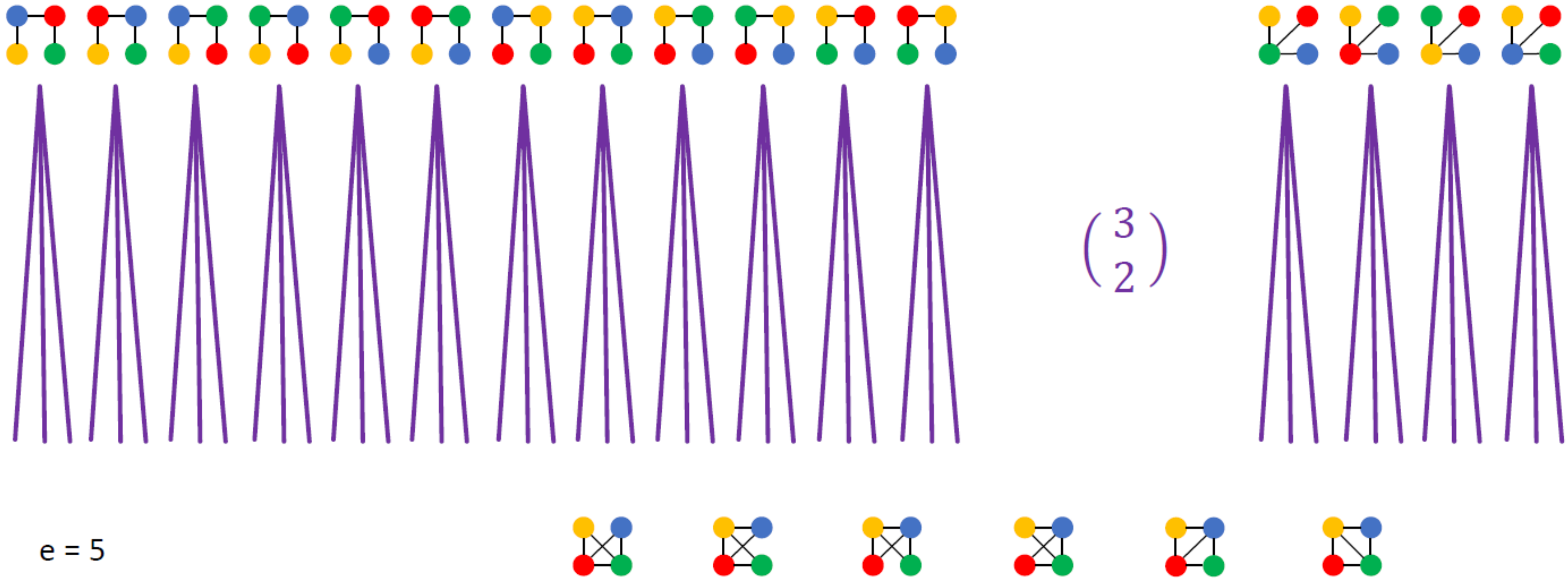
- Szádoczki, Zs., Bozóki, S. & Tekile, H. A. (2022). Filling in pattern designs for incomplete pairwise comparison matrices: (Quasi-)regular graphs with minimal diameter. Omega, 107(C), https://doi.org/10.1016/j.omega.2021.102557.

GRAPH of labelled graphs

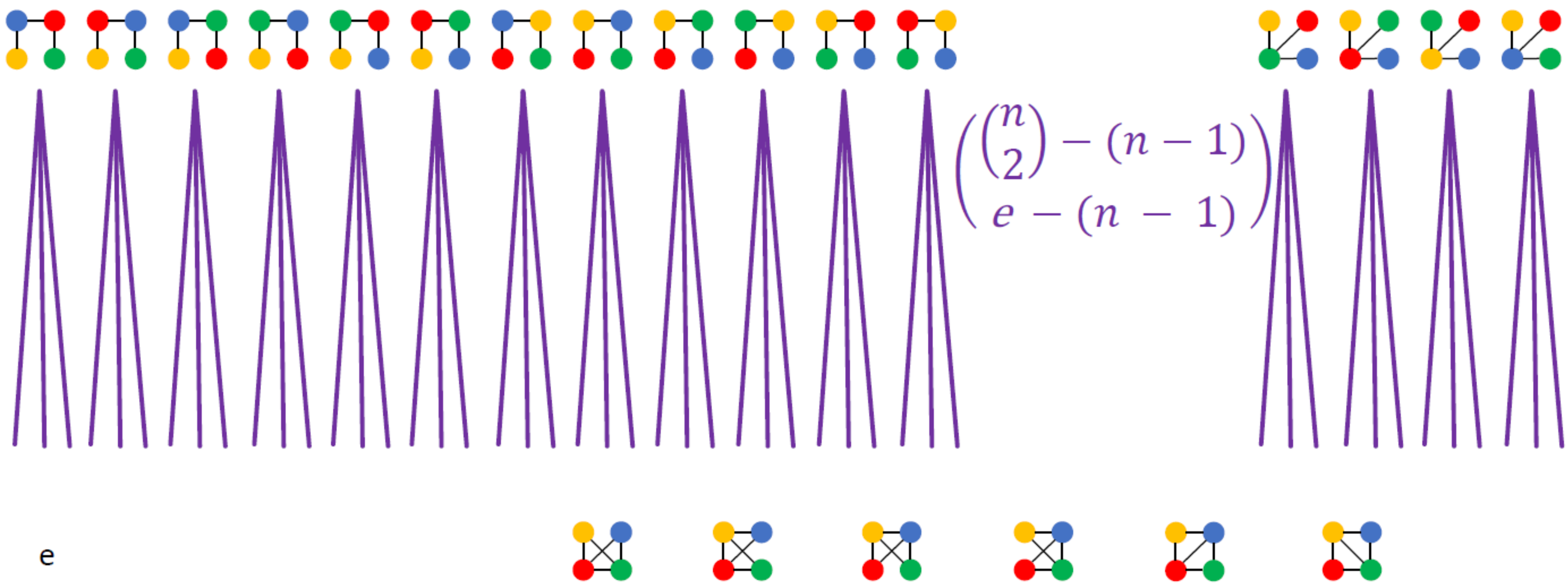




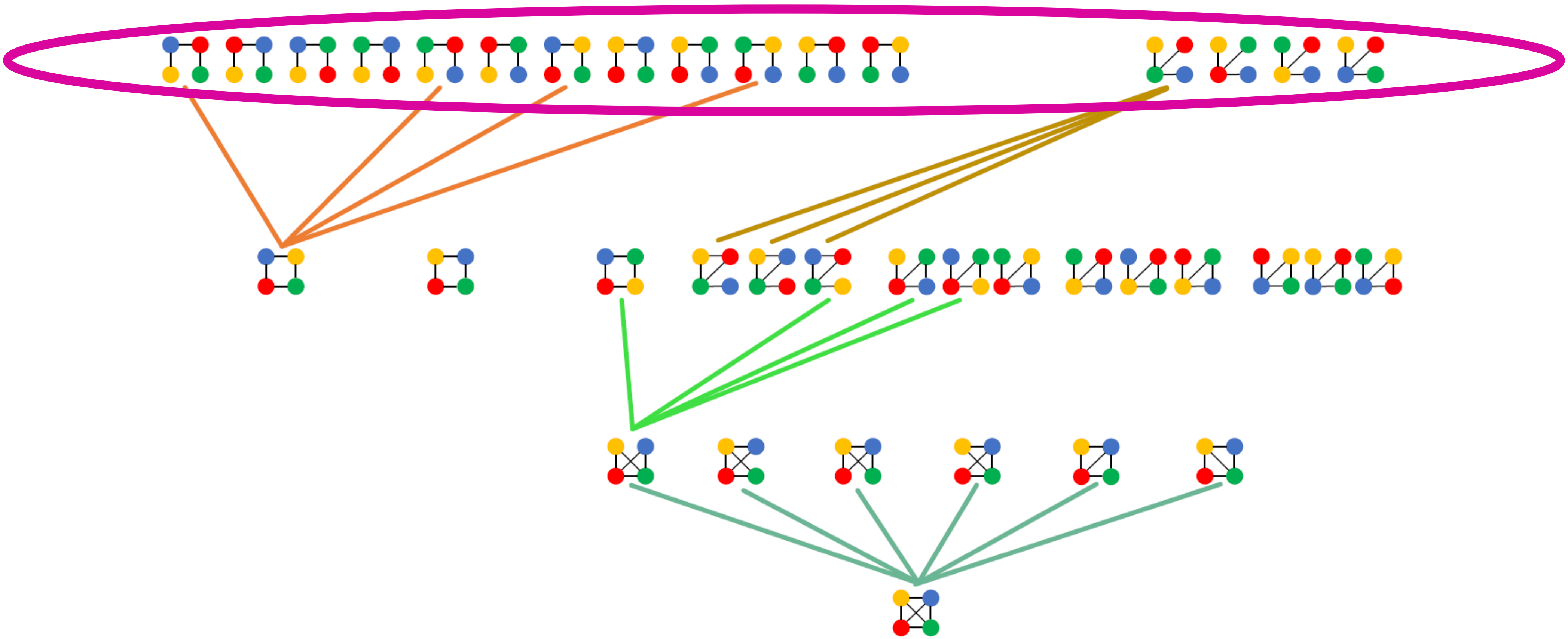
half regular bipartite GRAPH



half regular bipartite GRAPH

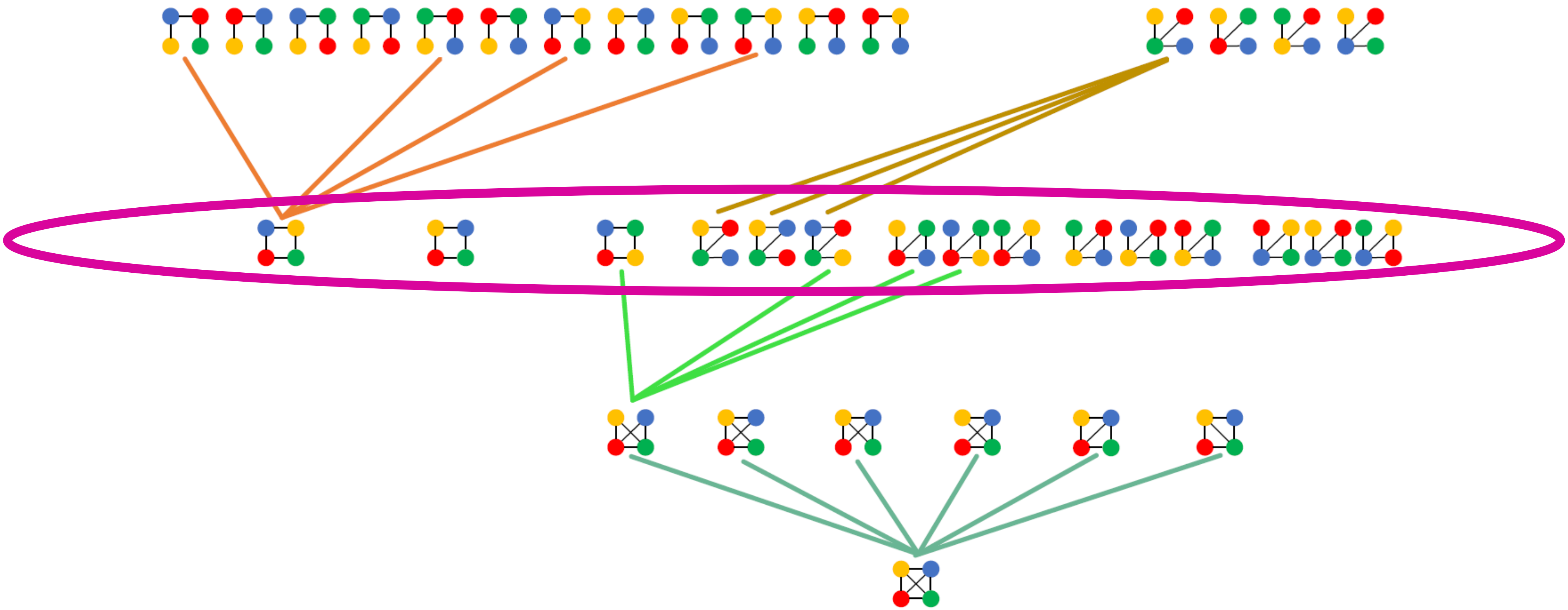


half regular bipartite GRAPH



Theorem

The geometric mean of weight vectors, calculated from all the incomplete matrices corresponding to the labelled connected subgraphs with *e* edges, is equal to the weight vector calculated from the complete matrix (with the logarithmic least squares method).



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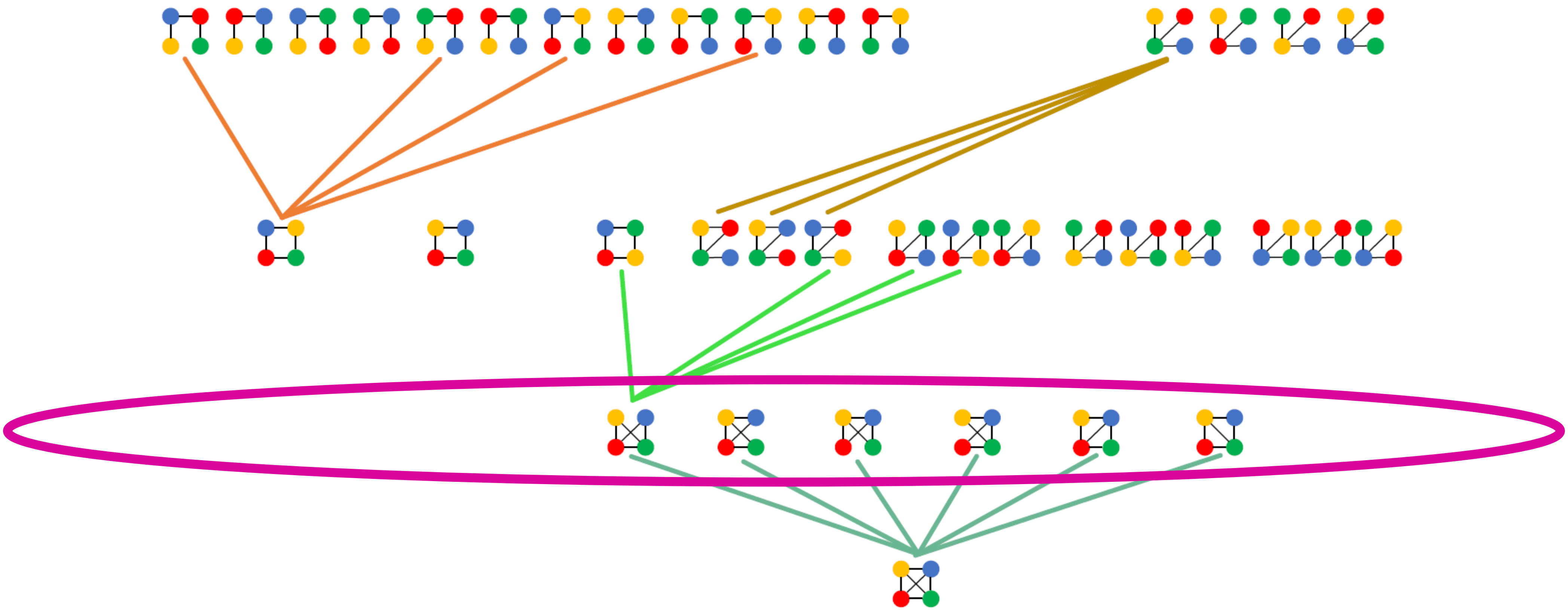

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