# Hamilton cycles in pseudorandom graphs

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- Finding sufficient conditions for graphs to contain a Hamilton cycle (a cycle touching every vertex) is a central theme in Combinatorics.

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  What about sparse graphs?

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- Threshold for $G(n,p)$ to be Hamiltonian is $p = \log n / n$ (Pósa 76')

- For all $d \geq 3$, the random $d$-regular graph $G_{n,d}$ is Hamiltonian whp.

(Bollobás 83', Fenner, Frieze 84', Robinson, Wormald 94', Cooper, Frieze, Reed 02', Krivelevich, Sudakov, Vu, Wormald 01')

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Def: An $n$-vertex graph is an $(n, d, \lambda)$-graph if it is $d$-regular and its eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ have $|\lambda_2|, |\lambda_n| \leq \lambda$. $\left(\lambda = \Omega(\sqrt{d})\right)$

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## Expander mixing lemma

$$(n, d, \lambda)\text{-graph } G \implies \forall A, B \subseteq V(G), \quad \left| e(A,B) - \frac{d|A||B|}{n} \right| \leq \lambda \sqrt{|A||B|}$$

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**Thm** (Krivelevich, Sudakov 03')

$$d/\lambda \geq 1000 \log n \cdot \frac{\log\log\log n}{(\log\log n)^2} \implies \text{Hamiltonicity}$$

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For all $\alpha > 0$, there is a constant $C = C(\alpha)$ such that if $d \geq n^{\alpha}$, then

$$d/\lambda > C \Rightarrow \text{Hamiltonicity}$$

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If $d \geq (\log n)^5$, then

$d/\lambda > C\log_d n \Rightarrow$ Hamiltonicity

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If $d \geq (\log n)^5$, then

$G$ has $\geq 1000\lambda n/d$ disjoint cycles $\implies$ Hamiltonicity

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Every connected Cayley graph has a Hamilton cycle.

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Every connected Cayley graph has a Hamilton cycle.

**Conj.** (Lovász 69')
Every connected vertex-transitive graph has a Hamilton cycle (except for five known examples)

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$\exists$ constant $C$ s.t. $\forall$ group $G$ and a set $S$ of $C \log n$ random elements, the Cayley graph $\Gamma(G, S)$ is Hamiltonian whp.

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(Christofides, Markström 08') If $S$ is a random set of elements in $G$, then $\Gamma(G,S)$ is an $(n,d,\lambda)$-graph with $\lambda \leq 100\sqrt{d\log n}$ (and $d = \Theta(|S|)$)

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- Our methods give a set $S$ with $|S| = O(\log n)$ but a fully combinatorial proof.

# An application with pol. dense graphs

## Additive patterns in multiplicative subgroups

Let $\mathbb{F}_q$ be a finite field and $A \subseteq \mathbb{F}_q$ a multiplicative subgroup of $\mathbb{F}_q^*$. How large does $A$ have to be for additive patterns to emerge?

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- (Alon, Bourgain 14') $\Omega(q^{1/3})$ is necessary for prime $q$

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Q (Alon, Bourgain 14'): When does there exist an ordering $x_1, \dots, x_{|A|}$ of $A$ s.t. $x_1+x_2, x_2+x_3, \dots, x_{|A|}+x_1 \in A$?

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- It can be shown that the graph with vertex set $A$ and edges $x \sim y$ if $x+y \in A$ is essentially an $\left(|A|, \frac{|A|^2}{q}, q^{1/2}\right)$-graph $\left(|A| \geq q^{3/4}\right)$

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- (Alon, Bourgain) $|A| \geq q^{3/4}\sqrt{\log q}$ is enough
- (Glock, M.C, Sudakov) $|A| \geq C q^{3/4}$ is enough

# Problem

Let $G$ be an $(n,d,\lambda)$-graph with $d/\lambda > C$ for some arb. large constant $C$.

Can you find a path of size longer than $n - \Theta\left(\frac{\lambda^2 n}{d^2}\right)$?

Pósa rotation

Thank you!