The 13th Combinatorics Days

Core-free Degrees of Toroidal Maps

Claudio Alexandre Piedade
Centro de Matemática da Universidade do Porto, Portugal
claudio.piedade@fc.up.pt
Maria Elisa Fernandes, Universidade de Aveiro, Portugal
maria.elisa@ua.pt

13th July, 2023

CMUP, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre s/n, 4169–007 Porto (Portugal). The author was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

Permutation Representations - What about them?

- Any finite group is isomorphic to a subgroup of S_n;

Permutation Representations - What about them?

- Any finite group is isomorphic to a subgroup of S_n;
- We can represent faithfully any finite group as a set of permutations on n points.

Permutation Representations - What about them?

- Any finite group is isomorphic to a subgroup of S_n;
- We can represent faithfully any finite group as a set of permutations on n points.
- Take the example of the following group

$$G := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = (\rho_0\rho_2)^2 = id_G \rangle$$

Permutation Representations - What about them?

- Any finite group is isomorphic to a subgroup of S_n;
- We can represent faithfully any finite group as a set of permutations on n points.
- Take the example of the following group

$$G := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = (\rho_0\rho_2)^2 = id_G \rangle$$

- $G \cong C_2 \times S_4$
- $G \to S_8$
- $\rho_0 = (1,2)(3,4)(5,6)(7,8); \rho_1 = (2,3)(6,7); \rho_2 = (3,5)(4,6);$

Permutation Representations - What about them?

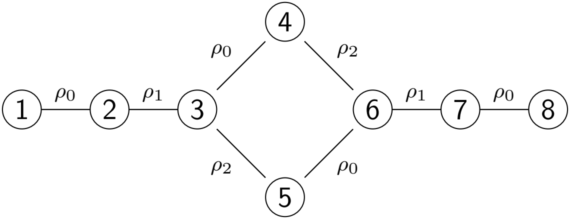
- Any finite group is isomorphic to a subgroup of S_n;
- We can represent faithfully any finite group as a set of permutations on n points.
- Take the example of the following group

$$G := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = (\rho_0\rho_2)^2 = id_G \rangle$$

- $G \cong C_2 \times S_4$
- $G \to S_8$
- $\rho_0 = (1,2)(3,4)(5,6)(7,8); \rho_1 = (2,3)(6,7); \rho_2 = (3,5)(4,6);$
- $G \to S_6$
- $\rho_0 = (1,2)(3,4)(5,6); \rho_1 = (2,3)(5,6); \rho_2 = (3,4)(5,6);$

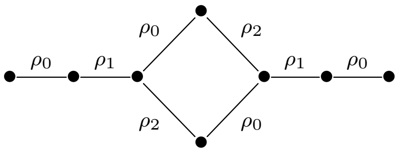
Faithful Permutation Representations Graph

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$



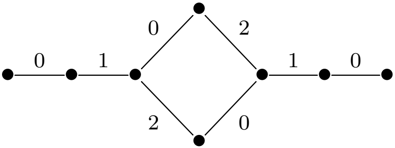
Faithful Permutation Representations Graph

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$



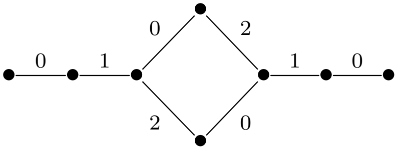
Faithful Permutation Representations Graph

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$



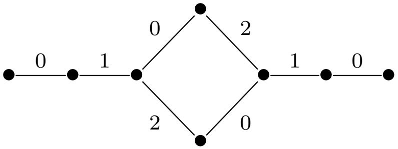
Faithful Permutation Representations - Transitive or Not?

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$

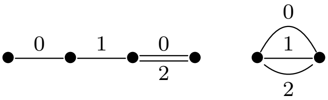


Faithful Permutation Representations - Transitive or Not?

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$

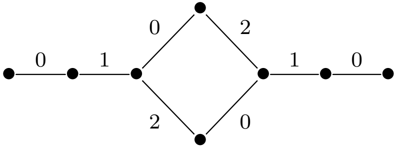


- $\rho_0 = (1,2)(3,4)(5,6)$
- $\rho_1 = (2,3)(5,6)$
- $\rho_2 = (3,4)(5,6)$

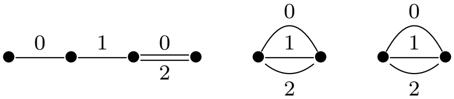


Faithful Permutation Representations - Transitive or Not?

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$

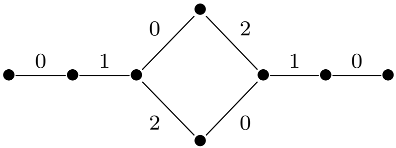


- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(5,6)(7,8)$
- $\rho_2 = (3,4)(5,6)(7,8)$

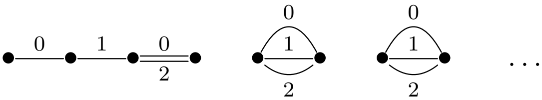


Faithful Permutation Representations - Transitive or Not?

- $\rho_0 = (1,2)(3,4)(5,6)(7,8)$
- $\rho_1 = (2,3)(6,7)$
- $\rho_2 = (3,5)(4,6)$



- $\rho_0 = (1,2)(3,4)(5,6)(7,8)(\ldots)$
- $\rho_1 = (2,3)(5,6)(7,8)(\ldots)$
- $\rho_2 = (3,4)(5,6)(7,8)(\ldots)$



Faithful Transitive Permutation Representations

- A permutation representation of a group gives the action of a group on a certain set of elements;
- Consider the left cosets of a subgroup H of G;

Faithful Transitive Permutation Representations

- A permutation representation of a group gives the action of a group on a certain set of elements;
- Consider the left cosets of a subgroup H of G;
- The action of G on these cosets give a permutation representation where the elements are cosets;

Faithful Transitive Permutation Representations

- A permutation representation of a group gives the action of a group on a certain set of elements;
- Consider the left cosets of a subgroup H of G;
- The action of G on these cosets give a permutation representation where the elements are cosets;
- The action is transitive;
- When is it faithful?

Faithful Transitive Permutation Representations

- A permutation representation of a group gives the action of a group on a certain set of elements;
- Consider the left cosets of a subgroup H of G;
- The action of G on these cosets give a permutation representation where the elements are cosets;
- The action is transitive;
- When is it faithful?
- G acts faithfully on the left cosets of H if and only if H is a core-free subgroup of G.

Core-free degrees

Definition (Core-free subgroup)

Let G be a group and $H \leq G$. We say H is a *core-free subgroup* of G if

$$\cap_{g \in G} H^g = \{id_G\}.$$

- The action of a group G on a core-free subgroup $H \leq G$ is always transitive and faithful, giving a faithful transitive permutation representation (FTPR) on the set of cosets G/H, with degree $|G : H|$.

Core-free degrees

Question

Given a group G, what is the set of possible indexes of core-free subgroups of G?

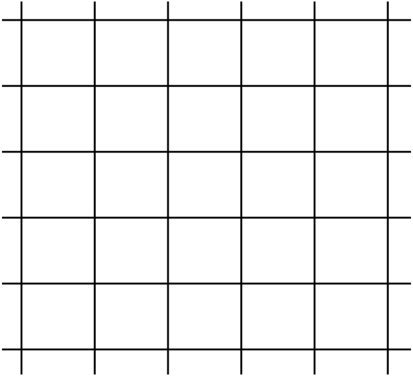
- For simple groups: All the index of their subgroups.
- Other groups, not so direct...

Definition (Degree of polytope/(hyper)map)

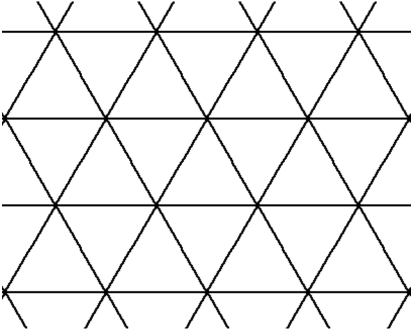
Let $\mathcal{P}$ be a polytope/(hyper)map. We say that n is a *degree of a polytope/(hyper)map* $\mathcal{P}$ if there is a core-free subgroup of the automorphism group of $\mathcal{P}$ with index n, i.e. there is a FTPR of $Aut(\mathcal{P})$ with degree n.

Coxeter groups for tesselations of the plane

Consider the infinite tesselations of the Euclidean plane by squares and triangles

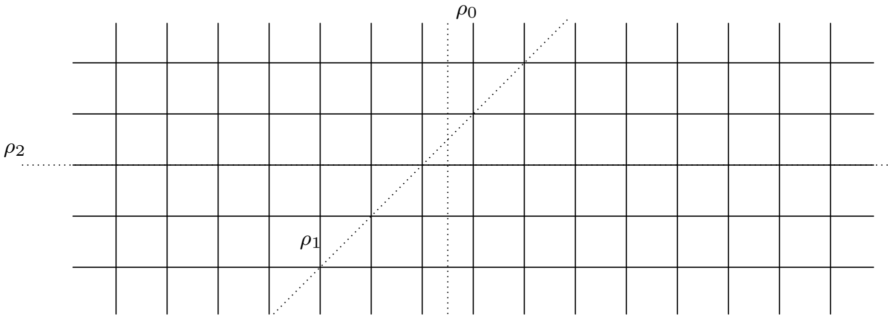


$[4, 4]$

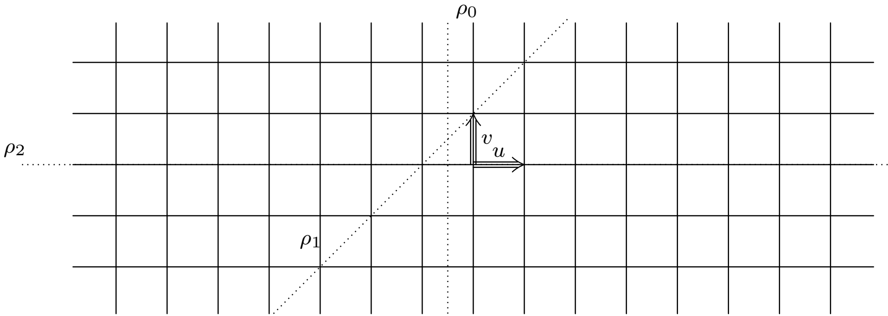


$[3, 6]$

Toroidal Map $\{4,4\}_{(s_1,s_2)}$

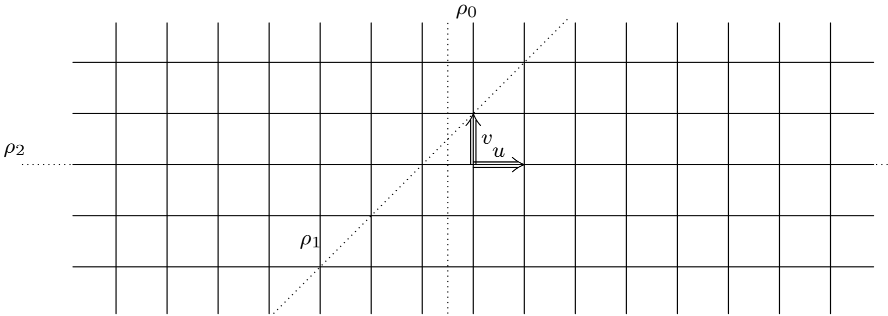


Toroidal Map $\{4,4\}_{(s_1,s_2)}$



$u = \rho_0\rho_1\rho_2\rho_1$
$v = u^{\rho_1}$
$T := \langle u, v \rangle$

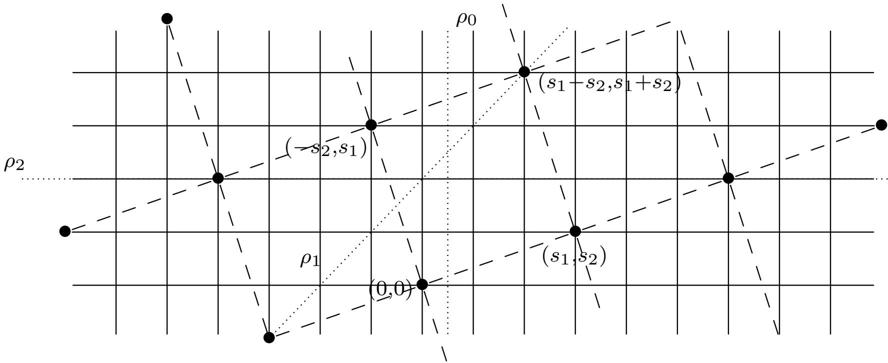
Toroidal Map $\{4,4\}_{(s_1,s_2)}$



$$u = \rho_0\rho_1\rho_2\rho_1$$
$$v = u^{\rho_1}$$
$$T := \langle u, v\rangle$$

$$[4,4]/\langle u^{s_1} v^{s_2}\rangle$$

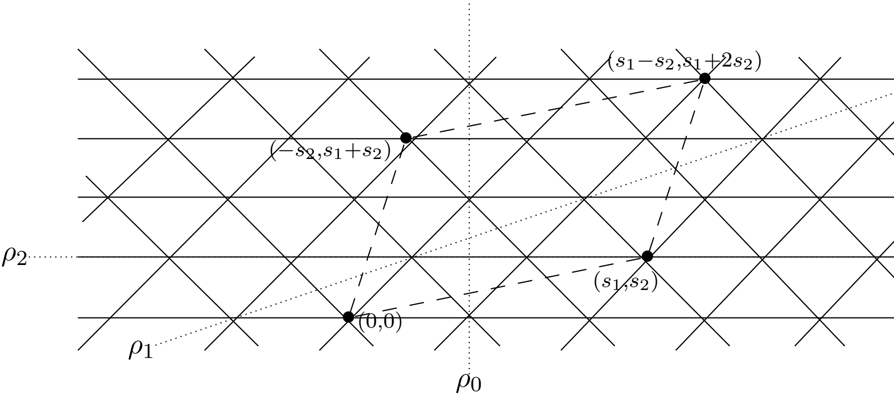
Toroidal Map $\{4,4\}_{(s_1,s_2)}$



$u = \rho_0\rho_1\rho_2\rho_1$
$v = u^{\rho_1}$
$T := \langle u, v\rangle$

$[4,4]/\langle u^{s_1} v^{s_2}\rangle$
Regular $\rightarrow s_1 s_2(s_1 - s_2) = 0 \rightarrow (s,0)$ or (s,s)
Chiral $\rightarrow s_1 s_2(s_1 - s_2) \neq 0$

Toroidal Map $\{3,6\}_{(s_1,s_2)}$



$u = \rho_0(\rho_1\rho_2)^2$

$v = u^{\rho_1}$

$T := \langle u, v\rangle$

$[3,6]/\langle u^{s_1}v^{s_2}\rangle$

Regular $\rightarrow s_1s_2(s_1-s_2) = 0 \rightarrow (s,0)$ or (s,s)

Chiral $\rightarrow s_1s_2(s_1-s_2) \neq 0$

Coxeter groups for tesselations of the plane

- We can quotient the Coxeter groups $[4,4]$ and $[3,6]$ by a translation subgroup and get the following groups:

$$[4,4]_{(s,0)} := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = \\ = (\rho_0\rho_1\rho_2\rho_1)^s = id_{[4,4]}\rangle$$

$$[4,4]_{(s,s)} := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = \\ = (\rho_0\rho_1\rho_2)^{2s} = id_{[4,4]}\rangle$$

$$[3,6]_{(s,0)} := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^6 = (\rho_0\rho_2)^2 = \\ = (\rho_0(\rho_1\rho_2)^2\rho_1)^s = id_{[3,6]}\rangle$$

$$[3,6]_{(s,s)} := \langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^6 = (\rho_0\rho_2)^2 = \\ = (\rho_0(\rho_1\rho_2)^2)^{2s} = id_{[3,6]}\rangle$$

Preliminary Results - Restrict to the $(s, 0)$

Conside the following:

- $G = \langle \rho_0, \rho_1, \rho_2 \rangle$ is the automorphism group of any toroidal maps $\{4,4\}_{(s,0)}$, $\{3,6\}_{(s,0)}$;
- $T = \langle u, v \rangle$ is the translation subgroup; Moreover $T \lhd G$ and is abelian (u and v commute);
- $o(u) = s$

Proposition

The translation subgroup T is isomorphic to $C_{o(u)} \times C_{gcd(s_1,s_2)}$.

Preliminary Results - Restrict to the $(s, 0)$

Conside the following:

- $G = \langle \rho_0, \rho_1, \rho_2 \rangle$ is the automorphism group of any toroidal maps $\{4,4\}_{(s,0)}$, $\{3,6\}_{(s,0)}$;
- $T = \langle u, v \rangle$ is the translation subgroup; Moreover $T \lhd G$ and is abelian (u and v commute);
- $o(u) = s$

Proposition

The translation subgroup T is isomorphic to $C_{o(u)} \times C_{gcd(s_1,s_2)}$.

Remark

If $(s, 0) \rightarrow T \cong C_s \times C_s$ *and* $|T| = s^2$
If (s, s), *as* $o(u) = 2s$, *then* $T \cong C_{2s} \times C_s$ *and* $|T| = 2s^2$

Preliminary Results - Restrict to the $(s, 0)$

- Suppose that there is a faithful transitive permutation representation of G with degree n.
- The translation subgroup T can either be transitive or intransitive. Since T is a normal subgroup of G, the T-orbits form a block system (which might be trivial).

Proposition

If T is transitive, then $n = |T| = s^2$.

Lemma

The size of a T-orbit is $k = o(u)d$ where d is a divisor of $gcd(s_1, s_2) = gcd(s, 0) = s$.

Preliminary Results

Proposition

Let G be a faithful transitive permutation representation of the rotational group of a toroidal (hyper)map with degree n. If $n \neq |T|$ then G is embedded into $S_k \wr S_m$ with $n = km$ $(m, k > 1)$ and we have

(i) $k = o(u)d = sd$ *where d is a divisor of s, and*

(ii) m *is a divisor of* $\frac{|G|}{|T|}$.

- For example, for the toroidal maps $\{4,4\}_{(s,0)}$, $|G| = 8s^2 = 8|T|$
- Hence,
  - if $m = 1$, then $k = |T| = s^2$
  - if $m \in \{2,4,8\}$, then $k = sd$, for some d divisor of s

Core-free Subgroups for the map $\{4,4\}_{(s,0)}$

For the toroidal maps $\{4,4\}_{(s,0)}$, remind that $o(u) = s$ and $|G| = 8|T| = 8s^2$.

Proposition

Let G be the automorphism group a toroidal map $\{4,4\}_{(s,0)}$, with $s > 2$, and let a, b such that $s = lcm(a,b)$. Then, the following subgroups (and their subgroups) are core-free:

1. $H = \langle \rho_0, \rho_1 \rangle$, *with index* $|G : H| = s^2$;
2. $H = \langle \rho_0 \rho_1 \rangle$, *with index* $|G : H| = 2s^2$;
3. $H = \langle \rho_0, \rho_2 \rangle$, *with index* $|G : H| = 2s^2$;
4. $H = \langle \rho_0 \rho_2 \rangle$, *with index* $|G : H| = 4s^2$;
5. $H = \langle id_G \rangle$, *with index* $|G : H| = 8s^2$;

Core-free Subgroups for the map $\{4,4\}_{(s,0)}$

For the toroidal maps $\{4,4\}_{(s,0)}$, remind that $o(u) = s$ and $|G| = 8|T| = 8s^2$.

Proposition

Let G be the automorphism group a toroidal map $\{4,4\}_{(s,0)}$, with $s > 2$, and let a, b such that $s = lcm(a,b)$. Then, the following subgroups (and their subgroups) are core-free:

1. $H = \langle u^a, v^b \rangle$, *with* $|G : H| = 8ab$;
2. $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0 \rangle$, *with* $|G : H| = 4ab$;
3. *If* $ab \neq s$, $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0, \rho_2 \rangle$, *with* $|G : H| = 2ab$;
4. $H = \langle u \rangle \rtimes \langle \rho_0, \rho_2 \rangle$, *with* $|G : H| = 2s$.

Remind that if $k = ds$. If $lcm(a,b) = s$, then there is a d divisor of s such that $ab = ds$.

Core-free Subgroups for the map $\{4,4\}_{(s,0)}$

1. $H = \langle u^a, v^b \rangle$, with $|G : H| = 8ab$;

Proof.

Suppose that $x \in H \cap H^{\rho_1} = \langle u^a, v^b \rangle \cap \langle u^b, v^a \rangle$. Then, since u and v commute, we have that $x = (u^a)^i (v^b)^j = (u^b)^k (v^a)^l$. Hence, we have that

$$ai \equiv bk \text{ mod } s$$

$$bj \equiv al \text{ mod } s.$$

Since ai is a multiple of both a and b, it is also a multiple of s and $ai \equiv 0$ mod s. The same reasoning can be used for bj, leading to $bj \equiv 0$ mod s. Hence, $x = id_G$ and H is core-free. The order of H is $\frac{s^2}{ab}$ thus $|G : H| = 8ab$. □

Core-free Subgroups for the maps $\{4, 4\}$

Theorem

Let G be the group of the toroidal maps $\{4,4\}_{(s_1,s_2)}$, and let d be a divisor of $gcd(s_1, s_2)$. Then n is a degree of G if and only if

- $(s_1, s_2) = (s, 0)$ *and*

$$n \in \left\{s^2,\ 2ds,\ 4ds,\ 8ds\right\};$$

Core-free Subgroups for the maps $\{4,4\}$

Theorem

Let G be the group of the toroidal maps $\{4,4\}_{(s_1,s_2)}$, and let d be a divisor of $gcd(s_1,s_2)$. Then n is a degree of G if and only if

- $(s_1,s_2) = (s,0)$ *and*

$$n \in \left\{ s^2,\ 2ds,\ 4ds,\ 8ds \right\};$$

- $(s_1,s_2) = (s,s)$ *and*

$$n \in \left\{ 2s^2,\ 4ds,\ 8ds,\ 16ds \right\};$$

Core-free Subgroups for the maps $\{4,4\}$

Theorem

Let G be the group of the toroidal maps $\{4,4\}_{(s_1,s_2)}$, and let d be a divisor of $gcd(s_1,s_2)$. Then n is a degree of G if and only if

- $(s_1,s_2)=(s,0)$ *and*

$$n \in \left\{s^2,\ 2ds,\ 4ds,\ 8ds\right\};$$

- $(s_1,s_2)=(s,s)$ *and*

$$n \in \left\{2s^2,\ 4ds,\ 8ds,\ 16ds\right\};$$

- *is chiral and*

$$n \in \left\{|T|,\ 2o(u)d,\ 4o(u)d\right\}.$$

Core-free Subgroups for the maps $\{3,6\}_{(s,0)}$

For the toroidal maps $\{3,6\}_{(s,0)}$, remind that $o(u) = s$ and $|G| = 12|T| = 12s^2$.

$$m \in \{1, 2, 3, 4, 6, 12\}$$

Proposition

Let G be the automorphism group of a toroidal map $\{3,6\}_{(s,0)}$. Then, the following subgroups (and their subgroups) are core-free:

1. $H = \langle \rho_1, \rho_2 \rangle$, *with index* $|G : H| = s^2$;
2. $H = \langle \rho_0, \rho_1 \rangle$, *with index* $|G : H| = 2s^2$;
3. $H = \langle \rho_0, \rho_2 \rangle$, *with index* $|G : H| = 3s^2$;
4. $H = \langle \rho_0 \rho_1 \rangle$, *with index* $|G : H| = 4s^2$;
5. $H = \langle \rho_0 \rho_2 \rangle$, *with index* $|G : H| = 6s^2$;
6. $H = \langle id_G \rangle$, *with index* $|G : H| = 12s^2$;

Core-free Subgroups for the maps $\{3,6\}_{(s,0)}$

For the toroidal maps $\{3,6\}_{(s,0)}$, remind that $o(u) = s$ and $|G| = 12|T| = 12s^2$.

$$m \in \{1,2,3,4,6,12\}$$

Proposition

Let G be the automorphism group of a toroidal map $\{3,6\}_{(s,0)}$. Then, the following subgroups (and their subgroups) are core-free:

1. $H = \langle u^d \rangle \rtimes \langle \rho_0, \rho_2 \rangle$, *with d divisor of s and $|G:H| = 3ds$;*
2. $H = \langle u^d \rangle \rtimes \langle \rho_0 \rho_2 \rangle$, *with d divisor of s and $|G:H| = 6ds$;*
3. $H = \langle u^a, v^b \rangle$, *with $s = lcm(a,b)$, and $|G:H| = 12ab$;*

Core-free Subgroups for the maps $\{3,6\}_{(s,0)}$

For the toroidal maps $\{3,6\}_{(s,0)}$, remind that $o(u)=s$ and $|G|=12|T|=12s^2$.

$$m \in \{1,2,3,4,6,12\}$$

Proposition

Let G be the automorphism group of a toroidal map $\{3,6\}_{(s,0)}$. Then, the following subgroups (and their subgroups) are core-free:

1. $H=\langle u^d\rangle \rtimes \langle \rho_0,\rho_2\rangle$, *with d divisor of s and $|G:H|=3ds$;*
2. $H=\langle u^d\rangle \rtimes \langle \rho_0\rho_2\rangle$, *with d divisor of s and $|G:H|=6ds$;*
3. $H=\langle u^a, v^b\rangle$, *with $s=lcm(a,b)$, and $|G:H|=12ab$;*
4. $H=\langle (v^{-\alpha}u)^d\rangle \rtimes \langle \rho_1\rho_2\rangle$, *with $|G:H|=2ds$;*
5. $H=\langle (v^{-\alpha}u)^d\rangle \rtimes \langle \rho_0\rho_1\rangle$, *with $|G:H|=4ds$.*

   with d divisor of s and α coprime of s/d such that $\alpha^2-\alpha+1\equiv 0 \bmod (s/d) \Leftrightarrow$ all prime divisors of s/d are 1 mod 6.

Core-free Subgroups for the maps $\{3, 6\}$

Theorem

Let G *be the group of the toroidal maps* $\{3, 6\}_{(s_1,s_2)}$*, and let* d *be a divisor of* $gcd(s_1, s_2)$*. Then* n *is a degree of* G *if and only if*

- $(s_1, s_2) = (s, 0)$ *and*

$$n \in \left\{s^2,\ 3ds,\ 6ds,\ 12ds\right\} \cup \left\{2d's,\ 4d's\right\};$$

Core-free Subgroups for the maps $\{3, 6\}$

Theorem

Let G be the group of the toroidal maps $\{3, 6\}_{(s_1, s_2)}$, and let d be a divisor of $gcd(s_1, s_2)$. Then n is a degree of G if and only if

- $(s_1, s_2) = (s, 0)$ *and*

$$n \in \left\{s^2,\ 3ds,\ 6ds,\ 12ds\right\} \cup \left\{2d's,\ 4d's\right\};$$

- $(s_1, s_2) = (s, s)$ *and*

$$n \in \left\{3s^2,\ 9ds,\ 18ds,\ 36ds\right\} \cup \left\{6d's,\ 12d's\right\};$$

Core-free Subgroups for the maps $\{3, 6\}$

Theorem

Let G be the group of the toroidal maps $\{3, 6\}_{(s_1,s_2)}$, and let d be a divisor of $gcd(s_1, s_2)$. Then n is a degree of G if and only if

- $(s_1, s_2) = (s, 0)$ *and*

$$n \in \left\{s^2,\ 3ds,\ 6ds,\ 12ds\right\} \cup \left\{2d's,\ 4d's\right\};$$

- $(s_1, s_2) = (s, s)$ *and*

$$n \in \left\{3s^2,\ 9ds,\ 18ds,\ 36ds\right\} \cup \left\{6d's,\ 12d's\right\};$$

- *is chiral and*

$$n \in \left\{|T|,\ 2|T|,\ 3o(u)d,\ 6o(u)d\right\}.$$

Core-free Degrees for the hypermaps $(3,3,3)$

Theorem

Let G be the group of the toroidal hypermap $(3,3,3)_{(s_1,s_2)}$, and let d be a divisor of $gcd(s_1,s_2)$. Then n is a degree of G if and only if

- $(s_1,s_2)=(s,0)$ *and*

$$n \in \left\{s^2,\ 3ds,\ 6ds,\ \right\} \cup \left\{2d's\right\};$$

- $(s_1,s_2)=(s,s)$ *and*

$$n \in \left\{3s^2,\ 9ds,\ 18ds\right\} \cup \left\{6d's\right\};$$

- *is chiral and*

$$n \in \left\{|T|,\ 3o(u)d\right\}.$$

Summary

We have a group G with a translation subgroup $T := \langle u, v \rangle$ such that $G = T \rtimes G_0$.
d is a divisor of $gcd(s_1, s_2)$, $gcd(s, 0) = gcd(s, s) = s$,
and d' is a divisor of s and all prime factors of s/d are $1 \ mod \ 6$

	Regular case (full group) $(s, 0)$ and (s, s)	Chiral case (rot. subgroup) (s_1, s_2)
$\{4, 4\}$	$\{\lvert T\rvert, 2o(u)d, 4o(u)d, 8o(u)d\}$	$\{\lvert T\rvert, 2o(u)d, 4o(u)d\}$
$\{3, 6\}$	$\{\lvert T\rvert, 2o(u)d', 3o(u)d, 4o(u)d', 6o(u)d, 12o(u)d\}$	$\{\lvert T\rvert, 2\lvert T\rvert, 3o(u)d, 6o(u)d\}$
$(3, 3, 3)$	$\{\lvert T\rvert, 2o(u)d', 3o(u)d, 6o(u)d\}$	$\{\lvert T\rvert, 3o(u)d\}$

What would be the next step?

- Classify core-free degrees of other groups!

And to do that...

And to do that...

I have developed a package for GAP: CoreFreeSub
https://github.com/CAPiedade/corefreesub

And to do that...

I have developed a package for GAP: CoreFreeSub
https://github.com/CAPiedade/corefreesub
Developed with Manuel Delgado (FCUP)

And to do that...

I have developed a package for GAP: CoreFreeSub
https://github.com/CAPiedade/corefreesub
Developed with Manuel Delgado (FCUP)
Let's take a look!

Acknowledgements

CMUP, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre s/n, 4169–007 Porto (Portugal). The author was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

The 13th Combinatorics Days

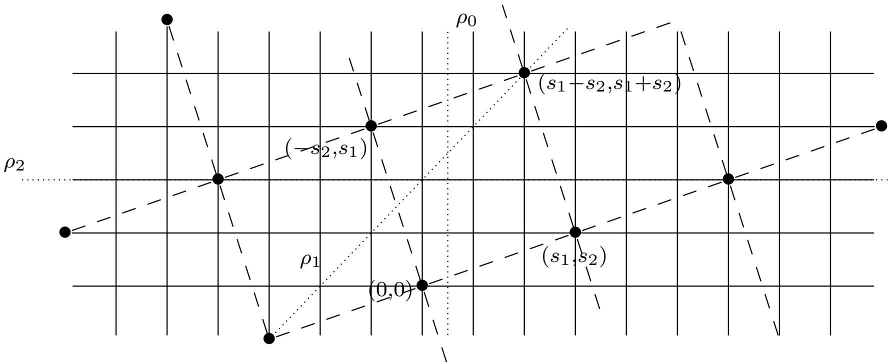
Core-free Degrees of Toroidal Maps

Claudio Alexandre Piedade
Centro de Matemática da Universidade do Porto, Portugal
claudio.piedade@fc.up.pt
Maria Elisa Fernandes, Universidade de Aveiro, Portugal
maria.elisa@ua.pt

13th July, 2023

CMUP, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre s/n, 4169–007 Porto (Portugal). The author was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

Toroidal Map $\{4,4\}_{(s_1,s_2)}$



back

Preliminary Results

Lemma

The size of a T-orbit is $k = o(u)d$ where d is a divisor of $gcd(s_1, s_2)$.

Proof.

Consider that σ and τ are the actions of the generators of T on a block of size k. Then $K := \langle\sigma, \tau\rangle$, $A := o(\sigma)$, $B := |K : \langle\sigma\rangle|$ and $C := |K : \langle\tau\rangle|$. We have that K has order AB and acts regularly on the block, hence $k = AB$. As σ and τ commute, we have the following
$K/\langle\sigma\rangle = \{\langle\sigma\rangle, \langle\sigma\rangle\tau, \langle\sigma\rangle\tau^2, \ldots, \langle\sigma\rangle\tau^{B-1}\}$
$K/\langle\tau\rangle = \{\langle\tau\rangle, \langle\tau\rangle\sigma, \langle\tau\rangle\sigma^2, \ldots, \langle\tau\rangle\sigma^{C-1}\}$.
Thus B divides $o(\tau)$ and C divides $o(\sigma) = A$. Let $D := A/C$. As $k = AB = o(\tau)C$ we have $o(\tau) = DB$. Now
$o(u) = lcm(o(\sigma), o(\tau)) = lcm(CD, BD) = D\,lcm(C, B)$ and
$k = AB = DCB = D\,lcm(C, B)\,gcd(C, B) = o(u)\,gcd(C, B)$.
To conclude the proof consider $d = gcd(C, B)$. It is easy to see that both B and C must be divisors of $gcd(s_1, s_2)$. Hence d must be a divisor of $gcd(s_1, s_2)$. □

Core-free Subgroups for the maps $\{3, 6\}_{(s_1,s_2)}$

Proposition

If $m = 2$ *then* $k = |T|$*.*

Proof.

The only possible permutation between blocks is with b.
Let $K = \langle u_1, v_1 \rangle$ be the action of T restricted to block $\mathcal{B}_1$.
As a fixes the blocks, we get $|u_1| = |v_1|$, implying that $|u_1| = |u|$.
Moreover, $|K : \langle u_1 \rangle| = |K : \langle v_1 \rangle| = d$, which is a divisor of $gcd(s_1, s_2)$.
Suppose there is a $j \in \{0, \ldots, o(u) - 1\}$ such that $u_1^d = v_1^j$.
Conjugating this by a, we have that $v_1^d = u_1^{d-j}$.
Moreover, conjugating $u_1^d = v_1^j$ by b, we get that $v_2^d = u_2^{d-j}$.
Finally, conjugating $v_1^d = u_1^{d-j}$ by b gives us that $u_2^d = u_2^{d-j} v_2^{j-d}$.
Substituting u_2^{d-j} by v_2^d, we get that $u_2^d = v_2^j$.
Hence, $u^d = v^j$.
Both d and j must be multiples of $gcd(s_1, s_2)$. Since d must divide $gcd(s_1, s_2)$, we get that $d = gcd(s_1, s_2)$. As $o(u) = \frac{|T|}{gcd(s_1,s_2)}$, then the size of the block is

$$k = o(u)d = \frac{|T|}{gcd(s_1,s_2)} gcd(s_1, s_2) = |T|$$