A local characterization of quasi-crystal graphs

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(joint work with Alan J. Cain, António Malheiro and Fátima Rodrigues)

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The 13th Combinatorics Days, Covilhã
July 13, 2023

This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

Motivation

Plactic monoid

[Lascoux, Schützenberger ’81]

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- Young tableaux, Schensted insertion

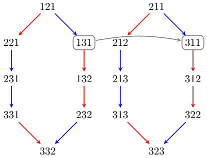
1	1	1	1
2	2		
3			

- Knuth relations

$$acb \equiv cab, \; a \leq b < c$$

$$bac \equiv bca, \; a < b \leq c$$

- Crystals



Motivation

Plactic monoid
[Lascoux, Schützenberger '81]

- Young tableaux, Schensted insertion

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

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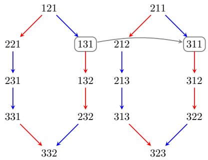

Hypoplactic monoid
[Krob, Thibon '97], [Novelli '00]

Motivation

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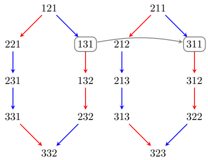


Hypoplactic monoid
[Krob, Thibon '97], [Novelli '00]

- Quasi-ribbon tableaux, Krob–Thibon insertion

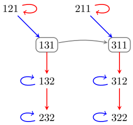
1	1			
	2	2	2	
			3	3

- Knuth + quartic relations

$$cadb \equiv acbd, \; a \leq b < c \leq d$$

$$bdac \equiv dbca, \; a < b \leq c < d$$

- Quasi-crystals



Crystals

Definition

A **crystal** of type A_{n-1} is a non-empty set $\mathcal{C}$ together with maps

$$\begin{aligned}
\tilde{e}_i, \tilde{f}_i &: \mathcal{C} \longrightarrow \mathcal{C} \sqcup \{\perp\} && \text{(Kashiwara operators)}\\
\tilde{\varepsilon}_i, \tilde{\varphi}_i &: \mathcal{C} \longrightarrow \mathbb{Z} \sqcup \{-\infty\} && \text{(length functions)}\\
wt &: \mathcal{C} \longrightarrow \mathbb{Z}^n && \text{(weight function)}
\end{aligned}$$

for $i \in I := \{1, \ldots, n-1\}$, satisfying the following:

C1. For any $x, y \in \mathcal{C}$, $\tilde{e}_i(x) = y$ iff $x = \tilde{f}_i(y)$, and in that case

$$wt(y) = wt(x) + \alpha_i, \quad \tilde{\varepsilon}_i(y) = \tilde{\varepsilon}_i(x) + 1, \quad \tilde{\varphi}_i(y) = \tilde{\varphi}_i(x) - 1$$

C2. $\tilde{\varphi}_i(x) = \tilde{\varepsilon}_i(x) + \langle wt(x), \alpha_i \rangle$

where $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$.

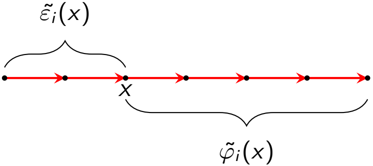
(This definition is generalized for other Cartan types)

Crystals

- A crystal is **seminormal** if

$$\tilde{\varepsilon}_i(x) = \max\{k : \tilde{e}_i(x)^k \neq \perp\}, \quad \tilde{\varphi}_i(x) = \max\{k : \tilde{f}_i(x)^k \neq \perp\},$$

for all $i \in I$ and $x \in \mathcal{C}$. In particular, $\tilde{\varepsilon}_i(x), \tilde{\varphi}_i(x) \geq 0$.



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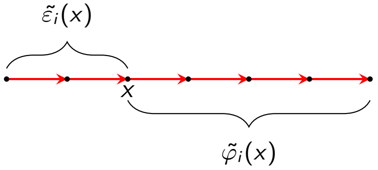

- To compute $\tilde{f}_i(w)$ and $\tilde{e}_i(w)$ on a word $w \in \{1 < \cdots < n\}^*$:
  - consider the subword with only symbols i and $i+1$, and cancel all pairs $(i+1)i$ (i-inversions), until there are no pairs left.
  - $\tilde{e}_i$ changes the *leftmost* $i+1$ to i, if possible; if not, it is $\perp$.
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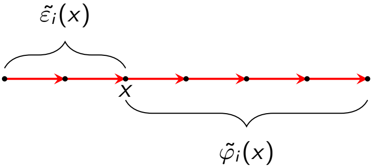

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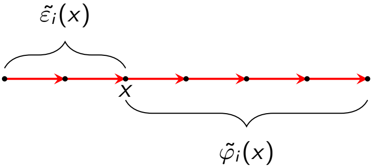

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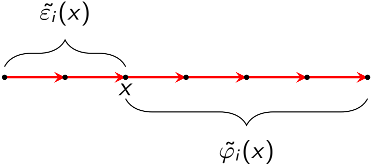

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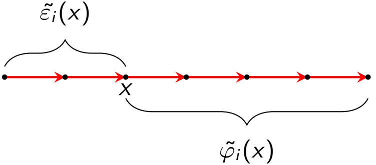

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$$\perp \xleftarrow{\tilde{e}_1} 1221111 \xleftarrow{\tilde{e}_1} \underline{1221112} \xrightarrow{\tilde{f}_1} 1221122 \xrightarrow{\tilde{f}_1} 2221122 \xrightarrow{\tilde{f}_1} \perp$$

Crystals

- The **crystal graph** associated to a crystal $\mathcal{C}$ is the directed weighted graph where $y \xrightarrow{i} x$ iff $\tilde{e}_i(x) = y$ iff $\tilde{f}_i(y) = x$.

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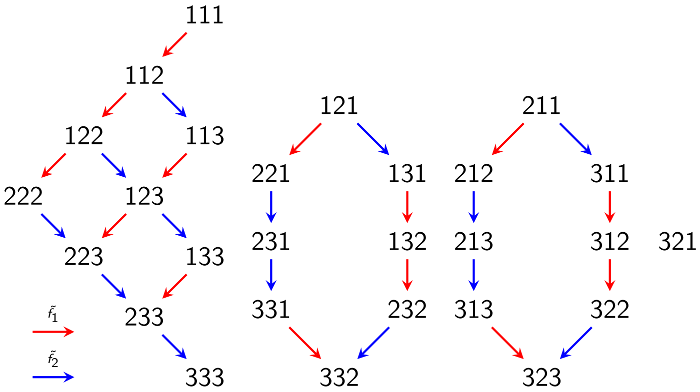


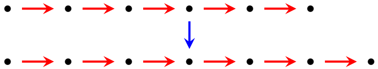
Stembridge crystals

- A **Stembridge crystal** is a seminormal crystal of simply-laced type that satisfies some local axioms [Stembridge '03]. These are the crystal graphs that correspond to representations of Lie algebras.
- The connected components have nice properties:
  - Unique highest weight element (source vertex), from which all vertices can be reached.
  - All components whose highest weight elements have the same weight are isomorphic.
  - In type A, the character of a connected component is a Schur function s_λ.

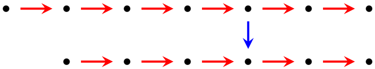
Stembridge crystals

Local axioms

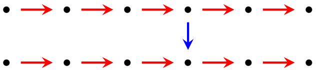
S1. If $\tilde{e}_i(x) = y$, then $\tilde{\varepsilon}_j(y)$ is equal to $\tilde{\varepsilon}_j(x)$ or $\tilde{\varepsilon}_j(x) + 1$ (the second case is possible only if $|i - j| = 1$).



or



for $|i - j| = 1$



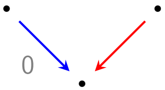
for $|i - j| > 1$

Stembridge crystals

Local axioms

S2. If $\tilde{e}_i(x) = y$ and $\tilde{e}_j(x) = z$, and $\tilde{\varepsilon}_i(z) = \tilde{\varepsilon}_i(x)$ then

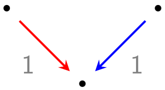
$$\tilde{e}_i\tilde{e}_j(x) = \tilde{e}_j\tilde{e}_i(x) \neq \perp.$$



(and dual axioms for $\tilde{f}_i$, $\tilde{f}_j$)

S3. If $\tilde{e}_i(x) = y$ and $\tilde{e}_j(x) = z$, and $\tilde{\varepsilon}_i(z) = \tilde{\varepsilon}_i(x) + 1$ and $\tilde{\varepsilon}_j(y) = \tilde{\varepsilon}_j(x) + 1$ then

$$\tilde{e}_i{\tilde{e}_j}^2\tilde{e}_i(x) = \tilde{e}_j{\tilde{e}_i}^2\tilde{e}_j(x) \neq \perp.$$



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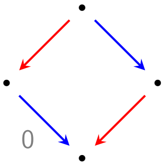

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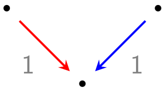

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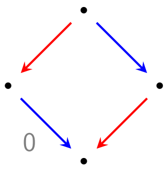

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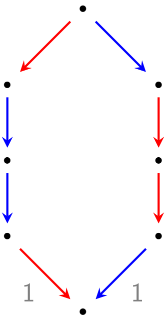

Quasi-crystals

- First introduced by Cain and Malheiro (2017), providing another characterization of the hypoplactic monoid of type A.
- Cain, Guilherme and Malheiro (2023) provided a definition of abstract quasi-crystals for other Cartan types.
- For type A, each connected component has a unique highest weight element, is isomorphic to a quasi-crystal of quasi-ribbon tableaux, and its character is a fundamental quasisymmetric function F_α.
- Recently, noting the decomposition of Schur functions into fundamental quasi-symmetric functions, Maas-Gariépy (2023) independently introduced quasi-crystals, as subgraphs of a connected component of a crystal graph.

Quasi-crystals

Definition (Cain, Guilherme, Malheiro '23)

A **quasi-crystal** of type A_{n-1} is a non-empty set $\mathcal{Q}$ together with maps

$$\begin{aligned} \ddot{e}_i, \ddot{f}_i &: \mathcal{Q} \longrightarrow \mathcal{Q} \sqcup \{\perp\} && \text{(quasi-Kashiwara operators)} \\ \ddot{\varepsilon}_i, \ddot{\varphi}_i &: \mathcal{Q} \longrightarrow \mathbb{Z} \sqcup \{-\infty, +\infty\} \\ wt &: \mathcal{Q} \longrightarrow \mathbb{Z}^n \end{aligned}$$

for $i \in \{1, \ldots, n-1\}$, satisfying the same axioms of crystals and an additional condition regarding $\ddot{\varepsilon}_i(x) = +\infty$.

- A quasi-crystal is **seminormal** if, for all $i \in I$ and $x \in \mathcal{Q}$,

  $$\ddot{\varepsilon}_i(x) = \max\{k : \ddot{e}_i(x)^k \neq \perp\}, \quad \ddot{\varphi}_i(x) = \max\{k : \ddot{f}_i(x)^k \neq \perp\}$$

  whenever $\ddot{\varepsilon}_i(x) \neq +\infty$.
- To compute $\ddot{f}_i(w)$ and $\ddot{e}_i(w)$ on a word $w \in \{1 < \cdots < n\}^*$:
  - If w has an i-inversion, $\ddot{f}_i(w) = \ddot{e}_i(w) = \perp$.
  - Otherwise, $\ddot{f}_i(w) = \tilde{f}_i(w)$ and $\ddot{e}_i(w) = \tilde{e}_i(w)$.

Quasi-crystals

The **quasi-crystal graph** associated to a quasi-crystal $\mathcal{Q}$ is the directed weighted graph where:

- $y \xrightarrow{i} x$ iff $\ddot{e}_i(x) = y$.
- x has an i-labelled loop iff $\ddot{\varepsilon}_i(x) = +\infty$ iff $\ddot{\varphi}_i(x) = +\infty$.

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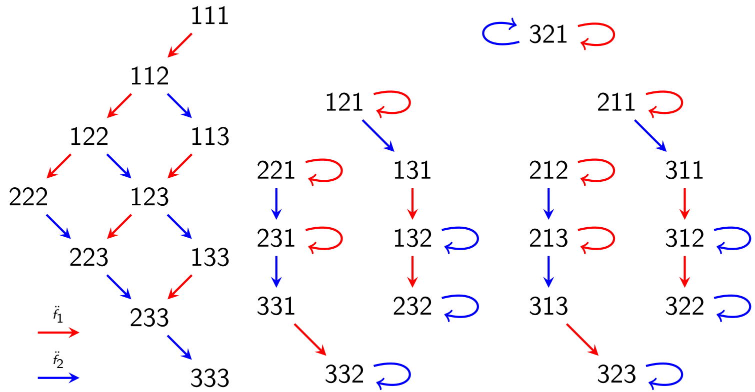

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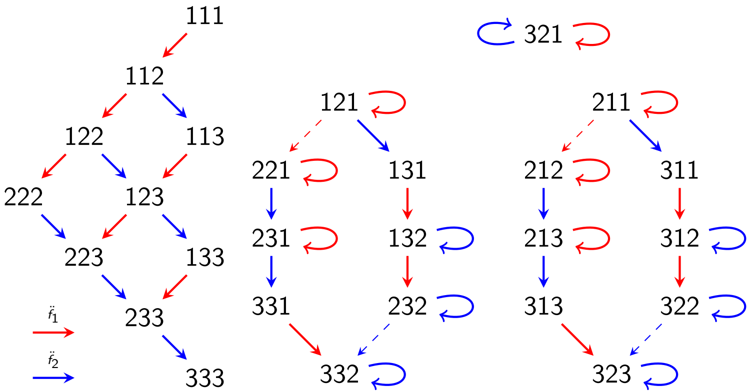

Local characterization of quasi-crystals

Local quasi-crystal axioms

LQC1. If $\ddot{e}_i(x) = y$, then:

- For $|i - j| > 1$, $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$.

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$$
\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet \arrow[d] \arrow[loop right] \\
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\quad \text{or} \quad
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\bullet \arrow[loop right] \arrow[d] \\
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- For $j = i + 1$,

$$\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y) \Leftrightarrow \left(\ddot{\varepsilon}_{i+1}(x) = +\infty \wedge \ddot{\varepsilon}_i(y) = 0\right) \Rightarrow \ddot{\varepsilon}_{i+1}(y) > 0.$$

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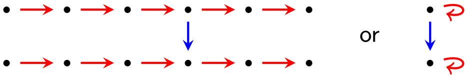

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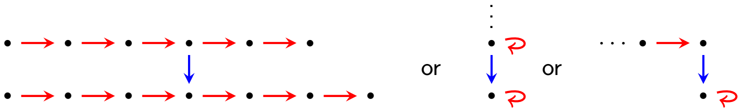

Local characterization of quasi-crystals

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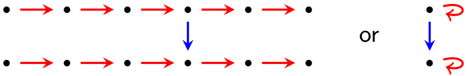

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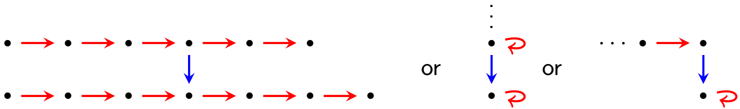
$$\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y) \Leftrightarrow \left(\ddot{\varepsilon}_{i+1}(x) = +\infty \wedge \ddot{\varepsilon}_i(y) = 0\right) \Rightarrow \ddot{\varepsilon}_{i+1}(y) > 0.$$



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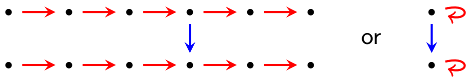
Local characterization of quasi-crystals

Local quasi-crystal axioms

LQC1. If $\ddot{e}_i(x) = y$, then:

- For $|i - j| > 1$, $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$.



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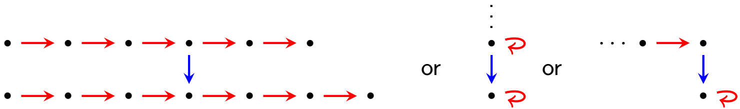

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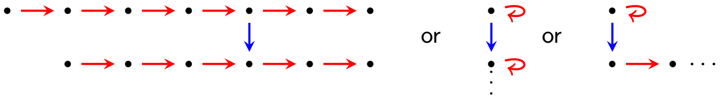
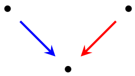


Local characterization of quasi-crystals

Local quasi-crystal axioms

LQC2. $\ddot{\varepsilon}_i(x) = 0$ iff $\ddot{\varphi}_{i+1}(x) = 0$, for $i \in \{1, \ldots, n-2\}$.

LQC3. If both $\ddot{e}_i(x)$ and $\ddot{e}_j(x)$ are defined, for $i \neq j$, then $\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp$ (and dual axiom for $\ddot{f}_i$, $\ddot{f}_j$.)



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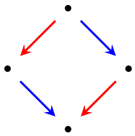

Local characterization of quasi-crystals

Theorem (Cain, Malheiro, Rodrigues, R. '23)

If $\mathcal{Q}$ is a quasi-crystal of type A (not necessarily seminormal) satisfying the local axioms, and such that $\ddot{\varepsilon}_i(x) \neq +\infty$ and $\ddot{\varphi}_i(x) \neq +\infty$, for all $i \in I, x \in \mathcal{Q}$, then $\mathcal{Q}$ is a weak Stembridge crystal (i.e. not necessarily seminormal).

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Theorem (Cain, Malheiro, Rodrigues, R. '23)

Let $\mathcal{Q}$ and $\mathcal{Q}'$ be connected components of seminormal quasi-crystal graphs of type A satisfying the local axioms, with highest weight elements u and v. If $wt(u) = wt(v)$, then there exists a weight-preserving isomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$.

Quasi-tensor product of quasi-crystals

- Cain, Guilherme, and Malheiro (2023) introduced a notion of quasi-tensor product of seminormal quasi-crystals, denoted $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$.
- $\mathcal{B}_n$ is the standard crystal of type A_{n-1}:

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-1} n$$

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$$\begin{tikzcd}
1 \mathbin{\ddot{\otimes}} 2 \arrow[loop left, red] \arrow[dr, blue] & \\
& 1 \mathbin{\ddot{\otimes}} 3 \arrow[dl, red] \\
2 \mathbin{\ddot{\otimes}} 3 \arrow[loop left, blue] &
\end{tikzcd}
\qquad\qquad
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A connected component of $\mathcal{B}_3 \mathbin{\ddot{\otimes}} \mathcal{B}_3$ | A connected component of $\mathcal{B}_3 \otimes \mathcal{B}_3$

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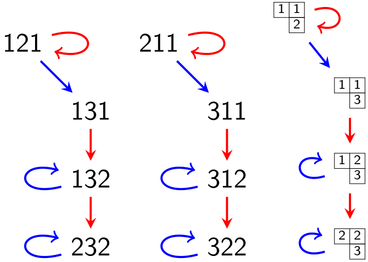
- The standard crystal $\mathcal{B}_n$ satisfies the local axioms.
- In particular, the quasi-crystal of words satisfies the local axioms.
- As a consequence, every connected component of a seminormal quasi-crystal satisfying the local axioms is isomorphic a quasi-crystal of quasi-ribbon tableaux.



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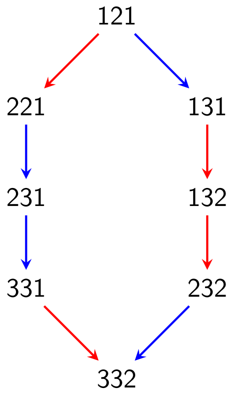

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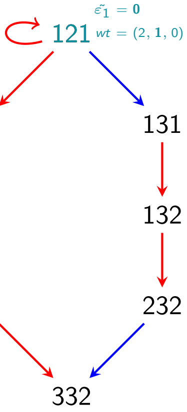

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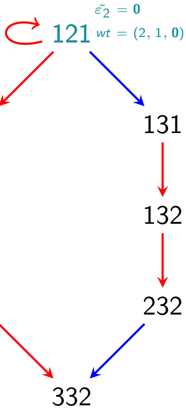

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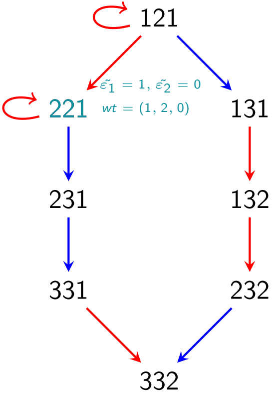

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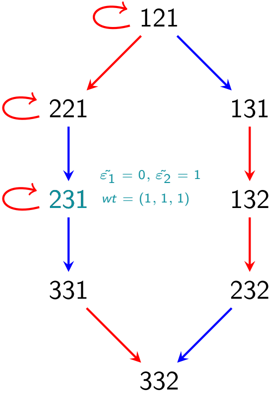

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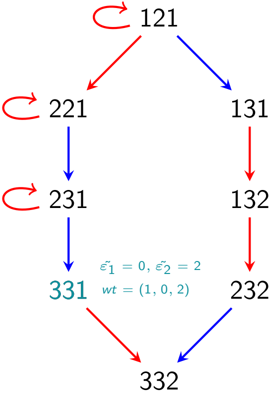

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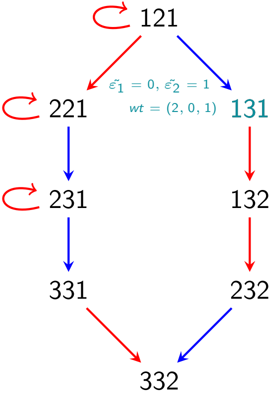

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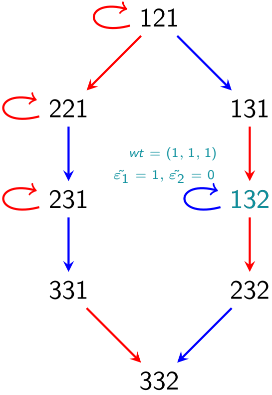

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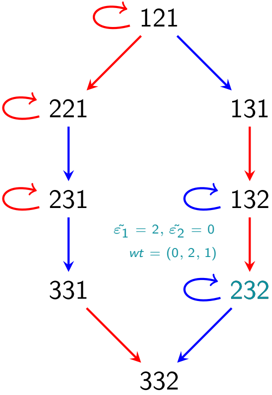

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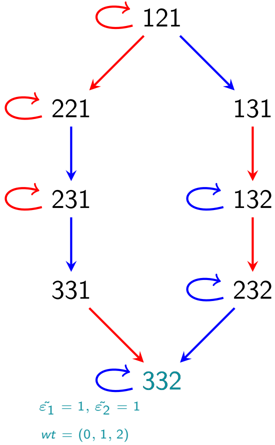

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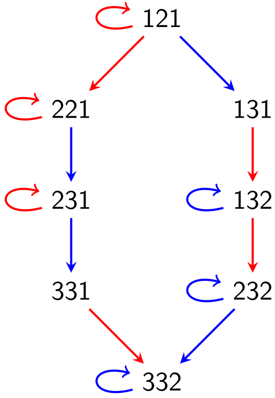

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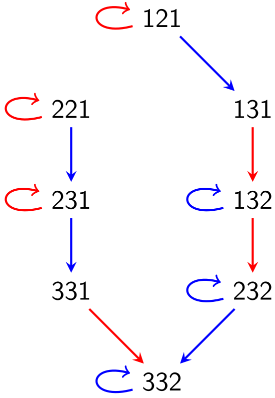

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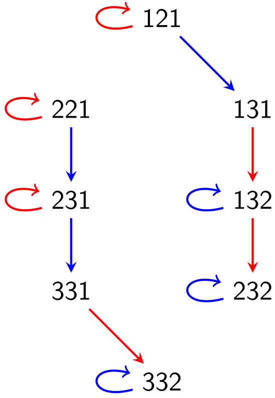

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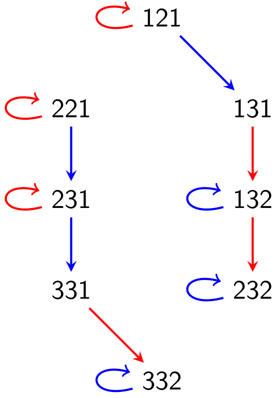
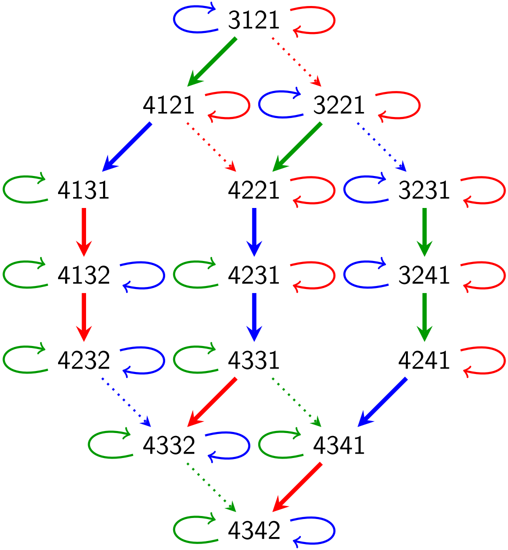


$$s_{21} = F_{21} + F_{12}.$$

From crystals to quasi-crystals



$$s_{311} = F_{311} + F_{131} + F_{113}$$

Some references

- A.J. Cain, R.P. Guilherme, A. Malheiro "Quasi-crystals for arbitrary root systems and associated generalizations of the hypoplactic monoid". arXiv:2301.00271.
- A.J. Cain, A. Malheiro "Crystallizing the hypoplactic monoid: from quasi-Kashiwara operators to the Robinson–Schensted–Knuth-type correspondence for quasi-ribbon tableaux", J. Algebr. Comb. **45** (2), 475–524 (2017).
- D. Krob, J.-Y. Thibon "Noncommutative symmetric functions. IV: Quantum linear groups and Hecke Algebras at $q = 0$", J. Algebr. Comb. **6** (4), 339 – 376 (1997).
- J.-C. Novelli "On the hypoplactic monoid", Discrete Math., **217** (1–3), 315–336 (2000).
- J.R. Stembridge "A local characterization of simply-laced crystals", Trans. Am. Math. Soc. **355** (12), 4807–4823 (2003).

Thank you!