On the generators of the Eulerian ideal of a graph

UC|UP Joint PhD Program in Mathematics

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of the University of Coimbra

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Graphs and homogeneous ideals

Simple graph $G = (V_G, E_G)$, with $V_G = \{1, \ldots, n\}$ and $\emptyset \neq E_G \subseteq \binom{V_G}{2}$.	$\longleftrightarrow$	Homogeneous ideals in polynomial rings: $\mathbb{K}[x_1, \ldots, x_n]$;

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- Toric ideal of a graph:

$$(t_{e_1} t_{e_3} \cdots t_{e_{2q-2}} - t_{e_2} t_{e_4} \cdots t_{e_{2q}} : e_1, e_2, \ldots, e_{2q} \text{ is a closed walk in } G).$$

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A generating set of $I(G)$ is

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The end!

Thank you for your attention !

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