A tour through spherical f-tilings

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• Summary
Introduction
We are interested in the study of a class of tilings of the sphere, spherical f-tilings.
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Spherical f-tilings are strongly related to the theory of isometric foldings:

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$$f : M \longrightarrow N$$

is an isometric folding if $f$ sends finite piecewise geodesic segments to finite piecewise geodesic segments of the same length.
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Let $\mathcal{F}(M, N)$ be set of all isometrics foldings from $M$ into $N$. Then,

1) $\mathcal{F}(M) = \mathcal{F}(M, M)$ is a semigroup with identity element $id_M$ and contains the isometry group $\mathcal{I}(M)$ as a sub-semigroup;
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2) for all $x, y \in M$, $d_N(f(x), f(y)) \leq d_M(x, y)$, where $d_M$ and $d_N$ are, respectively, the induced metrics on $M$ and $N$ by their Riemannian structure. And so any isometric folding is a continuous map;
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iii) any differentiable isometric folding is an isometry.
Isometric folding

\[ \Sigma f \] - The set of all **singularities** of \( f \) (points where \( f \) fails to be differentiable).
Isometric folding

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An isometric folding $f$ is said non trivial if $\Sigma f \neq \emptyset$.

A general description of $\Sigma f$, for any $f \in \mathcal{F}(M, N)$, was given by S. A. Robertson in 77.
Let $M$ and $N$ be complete Riemannian 2-manifolds and let $f \in \mathcal{F}(M, N)$. 
Description of $\Sigma f$ for surfaces

Let $M$ and $N$ be complete Riemannian 2-manifolds and let $f \in \mathcal{F}(M, N)$.

The singularities of $f$ near $x$ ($x \in \Sigma f$) form the image of an even number of geodesic rays emanating from $x$ and making alternated angles $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n$, where

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = \pi. \quad (1)$$
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$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = \pi. \tag{1}$$

The singularity set of an isometric folding on surfaces can be seen as an embedded graph of even valency at any vertex and satisfying the angle folding relation \((1)\).
Example of a planar isometric folding

\[ Re \]

\[ Id \]
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The set of singularities
Spherical folding tilings
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A spherical folding tiling is an edge-to-edge finite polygonal-tiling $\tau$ of $S^2$ whose underlying graph is of the type described in [1].
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We shall denote by $\mathcal{T}(S^2)$ the set of all folding tilings of $S^2$ identifying the singularity sets of non-trivial foldings with spherical folding tilings.
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  obs: The prototile must be a spherical triangle since any (convex) polyhedron in $\mathbb{R}^3$ must have, at least, a triangular face or a vertex of valency 3.
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Ten years later was established the complete classification of all triangular spherical monohedral tilings. (which obviously includes the monohedral f-tilings). Y. Ueno, Y. Agaoka - 2002.
Dihedral spherical f-tilings:

- Triangle + Triangle
- Triangle + Quadrangle
- Triangle + Convex Polygon
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Spherical folding tilings

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2.1 Dihedral f-tilings by spherical triangles and spherical squares.
Spherical folding tilings

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2.2 Dihedral f-tilings by spherical triangles and spherical rhombi.
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2.1 Dihedral f-tilings by spherical triangles and spherical squares.

2.2 Dihedral f-tilings by spherical triangles and spherical rhombi.

2.3 Dihedral f-tilings by spherical triangles and spherical parallelograms with distinct pairs of congruent opposite angles and with distinct pairs of congruent opposite sides.
3. Classification of all dihedral f-tilings of the sphere by triangles and $r$-sided regular polygons ($r \geq 5$), C. P. Avelino, A. F. Santos - 2008.
4. Classification of all dihedral triangular f-tilings of the sphere whose prototiles are an equilateral triangle and any other triangle, A. M. Breda, P. S. Ribeiro and A. F. Santos - from 2008 to 2009.
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5. Classification of all dihedral f-tilings of the sphere by isosceles trapezoids and (equilateral and isosceles) triangles, C. P. Avelino, A. F. Santos - 2011.
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Spherical folding tilings

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7. Classification of the dihedral f-tilings of the sphere by two right triangles being one of each isosceles, C. P. Avelino, A. F. Santos - 2012.
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7. Classification of the dihedral f-tilings of the sphere by two right triangles being one of each isosceles, C. P. Avelino, A. F. Santos - 2012.

8. Classification of the dihedral f-tilings of the sphere by any two isosceles triangles, A. M. Breda, P. S. Ribeiro - in work.
Dihedral f-tilings by isosceles triangles and isosceles trapezoids
Let $S^2$ be the Euclidean sphere of radius 1. A spherical isosceles trapezoid is a spherical quadrangle congruent to the intersection of two spherical lunes, $Q = L_1 \cap L_2$, where $L_1$ and $L_2$ have vertices in the plane $x = 0$, in orthogonal positions, and $L_1$ has the point $(1, 0, 0)$ at its center.
Adjacency cases

Let $\Omega(Q, T)$ be the set, up to an isomorphism, of all dihedral f-tilings of $S^2$ whose prototiles are an isosceles trapezoid $Q$ and an isosceles triangle $T$. 

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![Diagram](Image)

I

II

III

IV
Proposition 1. If $\Omega(Q, T) \neq \emptyset$ then there cannot be a pair of tiles as in cases I or IV.
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Proposition 2. Let $Q$ and $T$ be a spherical isosceles trapezoid and a spherical isosceles triangle, respectively, such that they are in adjacent positions as in case II. Then, $\Omega(Q, T) \neq \emptyset$ iff

(i) $\alpha_1 + \beta = \pi$ and $\alpha_2 = \gamma = \frac{\pi}{2}$ or

(ii) $\alpha_1 + \gamma = \pi$, $2\alpha_2 + \gamma = \pi$ and $\beta = \frac{\pi}{2}$ or

(iii) $\alpha_1 + \gamma = \pi$, $\alpha_2 = \frac{\pi}{3}$ and $\beta = \frac{\pi}{2}$ or

(iv) $\alpha_1 + \gamma = \pi$, $\alpha_2 = \frac{\pi}{2}$ and $\beta = \frac{\pi}{k}$, for some $k \geq 2$. 

Results

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Proposition 2. Let $Q$ and $T$ be a spherical isosceles trapezoid and a spherical isosceles triangle, respectively, such that they are in adjacent positions as in case II. Then, $\Omega(Q, T) \neq \emptyset$ iff

\begin{enumerate}
\item $\alpha_1 + \beta = \pi$ and $\alpha_2 = \gamma = \frac{\pi}{2}$
\end{enumerate}

Figure 1: $f$-tiling $T$
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$\alpha_1 + \gamma = \pi$, $2\alpha_2 + \gamma = \pi$ and $\beta = \frac{\pi}{2}$.

Figure 2: $t$-tiling $C$
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(iii) $\alpha_1 + \gamma = \pi$, $\alpha_2 = \frac{\pi}{3}$ and $\beta = \frac{\pi}{2}$.

Figure 3: $f$-tiling $\overline{C\gamma}$, $\gamma \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$
Proposition 1. If $\Omega(Q, T) \neq \emptyset$ then there cannot be a pair of tiles as in cases I or IV.

Proposition 2. Let $Q$ and $T$ be a spherical isosceles trapezoid and a spherical isosceles triangle, respectively, such that they are in adjacent positions as in case II. Then, $\Omega(Q, T) \neq \emptyset$ iff

$$(iv) \quad \alpha_1 + \gamma = \pi, \quad \alpha_2 = \frac{\pi}{2} \quad \text{and} \quad \beta = \frac{\pi}{k}, \quad \text{for some} \ k \geq 2:\$$

**Figure 4:** $f$-tiling $\mathcal{R}_{\alpha_1}^k$, $k \geq 2$ and $\alpha_1 \in \left(\frac{\pi}{2}, \frac{(k+1)\pi}{2k}\right)$
Proposition 3. If two spherical isosceles trapezoids are in adjacent positions as in case III, then \( \Omega(Q, T) = \{ \mathcal{R}_{\alpha_2}^k \mid k \geq 3 \} \cup \{ C \} \), where \( \mathcal{R}_{\alpha_2}^k \) is a dihedral f-tiling satisfying \( \alpha_1 + \gamma = \pi, \alpha_2 + \beta = \pi \) and \( \gamma = \frac{\pi}{k} \), with \( \alpha_2 \in \left( \pi - \arccos \left( - \cos^2 \frac{\pi}{k} \right), \frac{2\pi}{k} \right) \) and \( k \geq 3 \).
Proposition 3. If two spherical isosceles trapezoids are in adjacent positions as in case III, then \( \Omega(Q, T) = \{ \bar{R}^k_{\alpha_2} \mid k \geq 3 \} \cup \{ C \} \), where \( \bar{R}^k_{\alpha_2} \) is a dihedral f-tiling satisfying 
\[ \alpha_1 + \gamma = \pi, \alpha_2 + \beta = \pi \text{ and } \gamma = \frac{\pi}{k}, \text{ with } \alpha_2 \in \left( \pi - \arccos \left( -\cos^2 \frac{\pi}{k} \right), \frac{2\pi}{k} \right) \text{ and } k \geq 3. \]

Figure 5: f-tiling \( \bar{R}^k_{\alpha_2} \), \( k \geq 3 \) and \( \alpha_2 \in \left( \pi - \arccos \left( -\cos^2 \frac{\pi}{k} \right), \frac{2\pi}{k} \right) \)
How to obtain the f-tilings?

An “algorithm”...
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(i) Consider restrictions over $Q$ and $T$;
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(ii) common vertex of $Q$ and $T$ (adjacents);
How to obtain the f-tilings?

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(iii) list of possible vertices; remove those that do not allow the local configuration give rise to a global configuration;
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(iv) build a planar representation;
How to obtain the f-tilings?

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(v) complete analysis of all the angles and edges; study the congruence of them;
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(iii) list of possible vertices; remove those that do not allow the local configuration give rise to a global configuration;

(iv) build a planar representation;

(v) complete analysis of all the angles and edges; study the congruence of them;

(vi) build a 3D representation.
An example: f-tiling $C$
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An example: f-tiling $C$

Dihedral f-tilings by isosceles triangles and isosceles trapezoids

- Spherical isosceles trapezoid
- Adjacency cases
- Results
- An “algorithm”
- An example

Summary
An example: f-tiling $C$

$M = 24$

$N = 16$

$|V| = 3$

$G(\tau) = C_2 \times D_4$
Summary
Spherical folding tilings

Dihedral \( f \)-tilings by isosceles triangles and isosceles trapezoids

Summary

\[
\begin{array}{cccccccc}
\text{f-Tiling} & \alpha_1 & \alpha_2 & \gamma & \beta & |V| & M & N & G(\tau) \\
\hline
T & \arccos \frac{1-\sqrt{5}}{2} & \frac{\pi}{2} & \frac{\pi}{2} & \pi - \alpha_1 & 3 & 8 & 4 & C_2 \\
C & \arccos \frac{-2}{3} & \frac{\alpha_1}{2} & \pi - \alpha_1 & \frac{\pi}{2} & 3 & 24 & 16 & C_2 \times D_4 \\
\bar{C}_\gamma & \pi - \gamma & \frac{\pi}{3} & \left(\frac{\pi}{4}, \frac{\pi}{3}\right) & \frac{\pi}{2} & 3 & 24 & 24 & O_h \\
R_{\alpha_1}^k & \left(\frac{\pi}{2}, \frac{(k+1)\pi}{2k}\right) & \frac{\pi}{2} & \pi - \alpha_1 & \frac{\pi}{k} & 3 & 4k & 4k & C_2 \times D_{2k} \\
R_{\alpha_2}^k & \pi - \gamma & \left(\alpha_2^k, \frac{2\pi}{k}\right) & \frac{\pi}{k} & \pi - \alpha_2 & 3 & 4k & 2k & D_{2k} \\
\end{array}
\]

Table 1: Combinatorial Structure of the Dihedral \( f \)-Tilings of \( S^2 \) by Isosceles Trapezoids and Isosceles Triangles

- \( \alpha_2^k = \pi - \arccos \left(-\cos^2 \frac{\pi}{k}\right), \ k \geq 3; \)
- \( |V| \) is the number of distinct classes of congruent vertices;
- \( M \) and \( N \) are, respectively, the number of triangles congruent to \( T \) and the number of isosceles trapezoids congruent to \( Q \), used in the dihedral \( f \)-tilings;
- \( G(\tau) \) is the symmetry group of each tiling \( \tau \in \Omega (Q, T) \); by \( C_n \) we mean the cyclic group of order \( n; \) \( D_n \) is the dihedral group of order \( 2n; \) the octahedral group is \( O_h \cong C_2 \times S_4 \) (the symmetry group of the cube).
Thank you!